On the mod-p representations of unramified U(2,1)

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Abstract

Let E/F be an unramified quadratic extension of non-archimedean local fields of odd residue characteristic p and let G be the unitary group in three variable $U(2,1)(E/F)$. In this thesis, we explore the smooth representation theory of G over a field $\tilde{E}$ of characteristic p. The main results are as follows. Firstly, we have classified the simple modules of the pro-p Iwahori-Hecke algebra of G and described the so-called supersingular ones, which is one-dimensional character. Secondly, for the hyperspecial maximal compact open subgroup K_0 of G and any irreducible smooth representation σ of K_0, and for any non-zero $\lambda \in \tilde{E}$, we have determined the subquotients of $\mathrm{ind}_{K_0}^G \sigma/(T_\sigma - \lambda)$ by matching them precisely with the irreducible subquotients of principal series of G, where T_σ is some Hecke operator in the spherical Hecke algebra of G with respect to K_0 and σ. The latter result confirms a conjecture of Abdellatif. We also include several results aimed towards proving that supersingular representations of G are not finitely presented.

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1 Introduction

1.1 Introduction

Representations of p-adic groups over the complex numbers $\mathbb{C}$ have been developed into a rich and fruitful theory during the last forty years, since Robert Langlands announced his remarkable conjectures on automorphic forms of adelic groups. These conjectures could be viewed as far reaching non-abelian generalizations of local and global class field theory. One of them (the local version), which nowadays is usually known by the name 'Local Langlands correspondence', very roughly speaking, aims to interpret Galois representations $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to {}^LG$ in terms of smooth complex representations of G, for any reductive p-adic group G. There has been much significant progress on these conjectures in the last twenty years, by Harris-Taylor [HT01] & Henniart [Hen00] for p-adic GL_n, more recently by Arthur [Art13] for classical groups over p-adic fields.

However, it is also interesting to study the smooth representations over a field of positive characteristic and pursue a potential local Langlands correspondence. In this spirit, there is already great interest arising in recent years to study mod-l ($l \neq p$) representations ([Vig96], [Vig01]), and mod-p representations of p-adic groups ([BL95], [BL94], [Bre03]). An excellent summary on the current development (up to the summer of 2010) of mod-p representations (and many related topics) is given by Breuil in [Bre10]. Roughly speaking, there is already essential progress on the classification of irreducible admissible non-supersingular representations of a p-adic connective reductive group, mainly due to the work of Herzig [Her11], Abe [Abe13], Henniart–Vignéras [HV12], and their forthcoming joint work. But still very little is known for the so-called supersingular representations, for any group other than $GL_2(\mathbf{Q}_p)$. Due to such difficulty, the mod-p local Langlands correspondence is only known at present for the group $GL_2(\mathbf{Q}_p)$. But for the groups $SL_2(\mathbf{Q}_p)$ and $U(1,1)(\mathbf{Q}_{p^2}/\mathbf{Q}_p)$, there is already a semi-simple correspondence established, see [Abd11] and [Koz12].

This thesis is devoted to the study of the mod-p representations of the unitary group in three variables defined with respect to the unramified quadratic extension of a non-archimedean local field. In contrast to the recent method of Satake isomorphism developed by Florian Herzig [Her11] and Henniart–Vignéras [HV10], the approach in this thesis is mainly that of Barthel–Livné, where we follow their papers on $GL_2(F)$ [BL95], [BL94] in most aspects.

In section 1.2, we introduce the most used notations in this thesis. In section 1.4, we present the main results that have been obtained so far, chapter by chapter.

The intelligence debt of this thesis owed to those classical authors should be very clear to the readers. But it would be certainly the author's fault, if this thesis still contains any mistakes or inaccuracy.

1.2 Notations

Let F be a non-archimedean local field, with valuation ring $\mathfrak{o}_F$ and maximal ideal $\mathfrak{p}_F$. Let k_F be its residue field of characteristic p. Let q be the cardinality of k_F. Fix a separable closure F_s of F. Assume $p \neq 2$. Let E be the unramified and quadratic extension of F in F_s. We have similar notations $\mathfrak{o}_E$, $\mathfrak{p}_E$, k_E for E. Denote by E^1 (resp. k_E^1) the subgroup of $E^\times$ (resp. $k_E^\times$) consisting of elements of norm 1. Let ϖ_E be a prime element of E, lying in F. Given a 3-dimensional vector space V over E, we identify it with E^3 (the usual column space in three variables), by fixing a basis of V. We equip V with the non-degenerate Hermitian form h:

$$\mathrm{h}: \ V \times V \to E, \, (v_1, v_2) \mapsto \ v_1^{\mathrm{T}} \beta v_2^{\sigma}, v_1, v_2 \in V.$$

Here σ (we will write it as $-$ in the following) is the non-trivial element of $\mathrm{Gal}(E/F)$, and β is the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The unitary group G we are going to consider is the subgroup of $GL(3, E)$ whose elements fix the Hermitian form h:

$$G = \{g \in \mathrm{GL}(3, E) : \mathrm{h}(gv_1, gv_2) = \mathrm{h}(v_1, v_2), \text{for any } v_1, v_2 \ \in V\}.$$

Let B be the subgroup of upper triangular matrices of G, then $B = HN$, where N is the unipotent radical of B and H is the diagonal subgroup of G. Denote an element of the following form in N by $n(x, y)$:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix}$$

where $(x, y) \in E^2$ satisfies $x\bar{x} + y + \bar{y} = 0$.

Denote by N_k, for any $k \in \mathbb{Z}$, the subgroup of N consisting of $n(x, y)$ with $y \in \mathfrak{p}_E^k$.

Let $\triangle$ be the tree associated to G. Denote by X_0 the set of vertices on $\triangle$, which consists of $\mathfrak{o}_E$-lattices $\mathcal{L}$ in E^3, such that

$$\varpi_E \mathcal{L} \subset \mathcal{L}^* \subset \mathcal{L},$$

where $\mathcal{L}^*$ is the dual lattice of $\mathcal{L}$, under the Hermitian form h, i.e., $\mathcal{L}^* = \{v \in V : h(v, \mathcal{L}) \in \mathfrak{p}_E\}$.

Let $\mathbf{v}$, $\mathbf{v}'$ be two vertices in X_0 represented by $\mathcal{L}$ and $\mathcal{L}'$. The vertices $\mathbf{v}$ and $\mathbf{v}'$ are said to be adjacent, if:

$$\mathcal{L}' \subset \mathcal{L} \text{ or } \mathcal{L} \subset \mathcal{L}'.$$

Let $\{e_{-1}, e_0, e_1\}$ be the standard basis of E^3. We consider the following two lattices in E^3:

$$\mathcal{L}_0 = \mathfrak{o}_E e_{-1} \oplus \mathfrak{o}_E e_0 \oplus \mathfrak{o}_E e_1, \ \ \mathcal{L}_1 = \mathfrak{o}_E e_{-1} \oplus \mathfrak{o}_E e_0 \oplus \mathfrak{p}_E e_1.$$

Denote respectively by $\mathbf{v}_0, \mathbf{v}_1$ the two vertices represented by $\mathcal{L}_0$ and $\mathcal{L}_1$. They are then adjacent. The group G acts on X_0 in a natural way, and X_0 consists of two orbits, i.e.,

$$X_0 = \{G \cdot \mathbf{v}_0\} \cup \{G \cdot \mathbf{v}_1\}.$$

The vertices in $\triangle^1 := \{G \cdot \mathbf{v}_0\}$ is of period one, and that in $\triangle^2 := \{G \cdot \mathbf{v}_1\}$ is of period two.

Let K_0, K_1 be respectively the stabilizers of $\mathbf{v}_0, \mathbf{v}_1$ in G, and let α be the matrix

$$\begin{pmatrix} \varpi_E^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi_E \end{pmatrix},$$

and put $\beta' = \beta\alpha^{-1}$. The groups K_0 and K_1 could be described explicitly

$$K_0 = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E \end{pmatrix} \cap G, \ K_1 = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \cap G$$

For an integer $n \in \mathbb{Z}$, put $\mathbf{v}_{2n} = \alpha^n \mathbf{v}_0, \mathbf{v}_{2n+1} = \alpha^n \mathbf{v}_1$. These vertices together form a standard apartment in $\triangle$: $\{\mathbf{v}_n, n \in \mathbb{Z}\}$. A general edge in the standard apartment is $e_{2n,2n\pm1} = (\mathbf{v}_{2n}, \mathbf{v}_{2n\pm1})$, for an integer $n \in \mathbb{Z}$, i.e., an edge with origin $\mathbf{v}_{2n}$ and terminus $\mathbf{v}_{2n\pm1}$. Let I be the stabilizer of $e_{0,1}$ in G, i.e., the intersection of K_0 and K_1. It is the standard Iwahori subgroup of G consisting of matrices which are upper triangular mod $\mathfrak{p}_E$. Denote by I_1 the unique pro-p Sylow subgroup of I. Put $H_0 = I \cap H$, $H_1 = I_1 \cap H$.

We have introduced several subgroups of G, say B, N, N_k, I, I_1, and later on we will use the notations B', N', N'_k, I', I'_1 for their conjugate subgroups of G by the element β. Also, we use the notation $n'(x, y)$ for the element in N':

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & -\bar{x} & 1 \end{pmatrix}.$$

Denote by β_0 and β_1 respectively the following two matrices:

$$\begin{pmatrix} 0 & 0 & -t^{-1} \\ 0 & 1 & 0 \\ t & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -t^{-1}\varpi_E^{-1} \\ 0 & 1 & 0 \\ t\varpi_E & 0 & 0 \end{pmatrix},$$

where $t = [u]^{\frac{q+1}{2}}$, u is a generator of the cyclic group $k_E^\times$, and $[\cdot]$ denotes the Teichmüller lift. They lie respectively in K_0 and K_1, with determinant 1.

Denote by $\mathcal{N}$ the normalizer of H in G. For a character χ of H, denote by χ^s the character of H, which is the non-trivial conjugate of χ induced from the action of $\mathcal{N}$ on H.

We define the unitary group $\overline{G}(k_F) = U(2,1)(k_E/k_F)$ over the residue field k_F in the same manner as G. Denote by $\overline{H}(k_F)$ and $\overline{U}(k_F)$ respectively the diagonal and upper unipotent subgroup of $\overline{G}(k_F)$. For simplicity, sometimes we will write them $\overline{G}, \overline{H}, \overline{U}$, etc.

There is a natural reduction map from the group K_0 to $\overline{G}(k_F)$, which is surjective, and we denote the corresponding kernel subgroup by K_0^1.

As a character χ of I is trivial on I_1, and $I/I_1 \cong \overline{H}(k_F)$, we will usually identify characters of I and $\overline{H}(k_F)$.

Finally, we look at two fundamental equations over $\mathfrak{o}_E$. The first is

$$a + \bar{a} = 0, \ a \in \mathfrak{o}_E. \tag{1}$$

We find inductively a finite set (non-canonically) $L_1 = \{l_n, 0 \leq n \leq q-1\}$:

$l_0 = 0$, and for $n \geq 1$ take l_n to be any integer a satisfying (1) and $\mathfrak{p}_E \nmid a - l_j$ for all $j < n$.

The second is

$$b\bar{b} + a + \bar{a} = 0, \ a, b \in \mathfrak{o}_E. \tag{2}$$

Similarly, we find a finite set $L_2 = \{m_k, 0 \leq k \leq q^3 - 1\}$ (non-canonically) as follows:

$m_0 = (0,0)$, and for $k \geq 1$, set $m_k = (b_k, a_k)$, in which (b_k, a_k) satisfies (2) and at least one of the relations $\mathfrak{p}_E \nmid a_k - a_j$ and $\mathfrak{p}_E \nmid b_k - b_j$ holds for all $j < k$.

We don't assume any further operations on L_1 and L_2. Put $L_1^* = L_1 \setminus \{0\}$ and $L_2^* = L_2 \setminus \{(0,0)\}$.

Finally, we fix a field $\tilde{E}$ of characteristic p (not necessarily algebraically closed).

1.3 Preliminary facts

We record some basic decompositions for G and some subgroups.

Proposition 1.1. (1) $G = BK_i$, *for* $i = 0, 1$.
(2) $G = \cup_{l\geq 0} K_i \alpha^l K_i$, *for* $i = 0, 1$.
(3) $K_0 = I \cup I\beta I,\ K_1 = I \cup I\beta' I$.
(4) $I = (B \cap I) \cdot N_1' = N_1' \cdot (B \cap I)$.

Lemma 1.2. *For $y \in \mathfrak{p}_E^{-l}$ for some $l > 0$, we have a BK_0-decomposition: $\beta n(x, y) = n(\bar{y}^{-1}x, y^{-1})\alpha^{-l} \cdot k$, where k is some matrix in I.*

Proof. More explicitly, for $y \neq 0$, we have

$$\beta n(x, y) = n(\bar{y}^{-1}x, y^{-1}) \cdot \mathrm{diag}(\bar{y}^{-1}, -\bar{y}y^{-1}, y) \cdot n'(-\bar{y}^{-1}\bar{x}, y^{-1}). \tag{3}$$

□

Remark 1.3. *There is a natural isomorphism :*

$$E^\times / F^\times \cong E^1,$$

which is induced by the homomorphism $\nu : x \mapsto x\bar{x}^{-1}$, for $x \in E^\times$. By definition, the kernel of ν is just $F^\times$. Applying Hilbert 90[1] *to the quadratic extension E/F, ν is also surjective. Similarly, we have an isomorphism :*

$$k_E^\times / k_F^\times \cong k_E^1.$$

The natural homomorphism from $E^\times$ to $k_E^\times$ sending an element $\varpi_E^l x \in E^\times$ ($l \in \mathbb{Z}$, $x \in U_E$) to the image of x in $k_E^\times$ induces an isomorphism

$$E^\times / F^\times U_E^1 \cong k_E^\times / k_F^\times.$$

In all, there is a canonical quotient map from E^1 to k_E^1 :

[1] The usual multiplicative form of Hilbert 90 says that any element of norm 1 in a cyclic field extension L'/L is of the form $x \cdot \epsilon(x)^{-1}$, for some $x \in L'^\times$. Here ϵ is a generator of the Galois group of L'/L. In fact, it is almost directly to verify Hilbert 90 in the case that $L' = k_E$ and $L = k_F$. Assume $L' = E$ and $L = F$. Fix a root of unity η of order $q^2 - 1$ in $E^\times$, which satisfies some quadratic relation. Write an element $\mathfrak{e}$ of norm 1 in $E^\times$ as $a_1 \cdot \eta + b_1$, for some $a_1,\ b_1 \in \mathfrak{o}_F$. If $a_1 = 0$, then $\mathfrak{e} = \pm 1$ and $-1 = t\bar{t}^{-1}$. Otherwise, the element $\mathfrak{f} = \eta + (b_1 + 1)a_1^{-1}$ is as desired: $\mathfrak{e} = \mathfrak{f}\bar{\mathfrak{f}}^{-1}$.

$$E^1 \cong E^\times/F^\times \twoheadrightarrow E^\times/F^\times U_E^1 \cong k_E^\times/k_F^\times \cong k_E^1.$$

Hence, the characters of E^1 (resp. k_E^1) are naturally in bijection with the characters of $E^\times$ (resp. $k_E^\times$) which are trivial on $F^\times$ (resp. $k_F^\times$). From the quotient map above, we may view a character of k_E^1 as a character of E^1. However, a character of $E^\times$ which is trivial on $F^\times$ is also trivial on U_E^1, i.e., it is indeed a character of $E^\times/F^\times U_E^1$, as U_E^1 is a pro-p group. In summary, we may identify characters of E^1 and k_E^1.

Remark 1.4. *Let χ be a character of the group H_0. We write χ as $\chi_1 \otimes \chi_2$, i.e., $\chi(\mathrm{diag}(x, y, \bar{x}^{-1})) = \chi_1(x)\chi_2(y)$, for $\mathrm{diag}(x, y, \bar{x}^{-1}) \in H_0$, where χ_1 and χ_2 are respectively characters of $k_E^\times$ and k_E^1. Then, it is immediate to check that $\chi = \chi^s$ is equivalent to χ_1 being trivial on the group $k_F^\times$; by Remark 1.3, it is equivalent to the existence of a unique character χ_1' of k_E^1, such that $\chi_1(x) = \chi_1'(x\bar{x}^{-1})$. Furthermore, χ factors through the determinant if and only if $\chi_2 = \chi_1'$. We will use this remark in several places later, especially the existence of χ_1', for a character $\chi = \chi_1 \otimes \chi_2$ such that $\chi = \chi^s$.*

1.4 Presentation of main results

We now describe our main results, where the notations and terminologies are mainly those introduced in 1.2.

1.4.1 The pro-p Iwahori-Hecke algebra of G and its simple modules

This is the content of chapter 2. We describe the basic structure of the pro-p Iwahori-Hecke algebra $\mathscr{H}_{I_1} := \mathrm{End}_G(\mathrm{ind}_{I_1}^G 1)$ of G and determine its simple modules explicitly (Proposition 2.26, Theorem 2.30).

We briefly mention what we have achieved in this chapter. We mainly follow the method in [Vig04]. As $\mathscr{H}_{I_1}$ is the direct sum of $\mathscr{H}(I, \chi) := \mathrm{End}_G(\mathrm{ind}_I^G \chi)$, for characters χ of I satisfying $\chi = \chi^s$, and $\mathscr{H}(I, \chi \oplus \chi^s) := \mathrm{End}_G(\mathrm{ind}_I^G \chi \oplus \chi^s)$, for χ satisfying $\chi \neq \chi^s$, we are led to investigate the structures of the Iwahori-Hecke algebras $\mathscr{H}(I, \chi)$ for all χ and classify their simple modules (Proposition 2.6, 2.9, and Proposition 2.16, Proposition 2.22, 2.23).

We also calculate explicitly the natural right action of $\mathscr{H}_{I_1}$ on the I_1-invariants of any principal series representation $\mathrm{ind}_B^G \varepsilon$ (Proposition 2.31), and by excluding them we give the definition of supersingular character of $\mathscr{H}_{I_1}$ (Definition 2.33), which are then exactly those simple modules of $\mathscr{H}_{I_1}$ for which some fixed central element acts as zero.

1.4.2 The compactly induced representation $\mathrm{ind}_{K_0}^G\sigma$

This is the content of chapter 3. We are mainly concerned with some initial properties of the compactly induced representation $\mathrm{ind}_{K_0}^G\sigma$, for an irreducible smooth representation σ of K_0, and the corresponding spherical Hecke algebra $\mathscr{H}(K_0,\sigma)$.

It is known that the I_1-invariants of σ is one-dimensional ([CL76], Corollary 6.5), and we fix a basis $\{v_0\}$ of it once and for all. The Iwahori subgroup I acts on σ^{I_1} as a character, and denote it by χ_σ.

We have first the following, which is indeed a special case of a general result due to Herzig [Her11], Henniart-Vignéras [HV10]. Our approach is that of [BL94], i.e., we compute explicitly the convolution relations which define the multiplicative operation of the algebra.

Corollary 1.5. *$\mathscr{H}(K_0,\sigma)$ is isomorphic to the polynomial ring in one variable $\tilde{E}[T]$, for some $T\in\mathscr{H}(K_0,\sigma)$.*

We next describe the action of $\mathscr{H}_{I_1}$ on the I_1-invariants of $\mathrm{ind}_{K_0}^G\sigma$; more specifically the right action of $\mathscr{H}(I,\chi)$ on the (I,χ)-isotypic subspace of $\mathrm{ind}_{K_0}^G\sigma$, where $\chi=\chi_\sigma$ or χ_σ^s. The space of I_1-invariants of $\mathrm{ind}_{K_0}^G\sigma$ is easily described, and it has a canonical basis $\{f_n,\ n\in\mathbb{Z}\}$ (3.2, (8)) (up to a scalar). We then describe clearly the natural action of $\mathscr{H}(I,\chi)$ on that space (Proposition 3.9, 3.10).

Before stating the result, we remind the reader that the structure of the I_1-invariants of a compact induction as a module over the pro-p Iwahori-Hecke algebra has been determined in general (for F-split groups) by Rachel Ollivier in her recent work on inverse Satake [Oll12].

As a by-product, we have the following key proposition.

Proposition 1.6. *Let σ be an irreducible smooth representations of K_0. Let $\chi=\chi_\sigma$ or χ_σ^s. Then, any non-zero $\mathscr{H}(I,\chi)$-submodule of $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$ is of finite co-dimension as a vector space.*

We then move to a further consideration of the Hecke operator T in Corollary 1.5. In Lemma 3.12 and Proposition 3.13, we determine $T[Id,v]$ explicitly, where $[Id,v]$ is the function in $\mathrm{ind}_{K_0}^G\sigma$, supported on K_0 and taking value v at Id.

As an application, we show

Proposition 1.7. *$\mathrm{ind}_{K_0}^G\sigma$ is free over $\mathscr{H}(K_0,\sigma)$ if σ is a character or a twist of St by a character. Here, St is the inflation to K_0 of the Steinberg representation of $\overline{G}(k_F)$.*

Although it is not known to us whether the above Corollary holds or not for general σ [2], one in any case has the following compromise:

Proposition 1.8. *For an irreducible smooth representation σ of K_0, the compactly induced representation $\mathrm{ind}_{K_0}^G \sigma$ is faithfully flat over the algebra $\mathscr{H}(K_0, \sigma)$.*

We remark that Elmar Große-Klönne [GK10] has studied such a topic in detail for a general F-split group, where he has obtained a sufficient condition for the universal module to be free over the corresponding spherical Hecke algebra. But it is not clear to the author whether his result can be extended to our situation or not.

We finally mention the next two results, which pave the way for some definitions in chapter 4.

The following Lemma has already been proved in [Abd11].

Lemma 1.9. *For a character ε of B and an irreducible smooth representation σ of K_0, the space $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma, \ \mathrm{ind}_B^G \varepsilon)$ is at most one-dimensional, and it is non-zero if and only if*

$$\varepsilon_0 = \chi_\sigma^s,$$

where ε_0 is the restriction of ε to H_0.

Proposition 1.10. *The Hecke operator T acts on the one-dimensional space $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma, \ \mathrm{ind}_B^G \varepsilon)$ as a scalar c_ε, which is given by*

$$c_\varepsilon = \varepsilon(\alpha) + \sum\nolimits_{y_1 \in k_E^\times;\ y_1 + \bar{y}_1 = 0} \ \varepsilon(-y_1^{-1}, 1, y_1).$$

Some refined descriptions of the Bruhat-Tits tree of G are also included in chapter 3; in particular we have put the definition of height and antecedent (Definition 3.31) in general. Lemma 3.26 is crucial and will be used essentially in a major argument of chapter 4. We also record some observations about the actions of I_1 on the tree, but they are not used anywhere else in this thesis.

1.4.3 A parametrization theorem

This is the content of chapter 4, which is a major part of this thesis. Theoretically, one wants to establish the equivalence between supercuspidal

[2] But under some natural hypothesis we have indeed verified it in general, see Assumption 3.20 and Proposition 3.22.

representations and supersingular representations, and the main result presented below should serve as a main ingredient towards that (see also the remark after Corollary 1.13). We remind the readers that in a forthcoming paper of Abe–Henniart–Herzig–Vignéras, they will show admissible supersingular representations are equivalent to admissible supercuspidal representations, for any p-adic connected reductive group. Our approach is again mainly that of Barthel–Livné, but there are indeed some technical differences in our case to carry out the tree argument. We address a little more on that at the beginning of chapter 4.

Theorem 1.11. *Assume $\tilde{E}$ is algebraically closed. Let π be an irreducible smooth representation of G and σ be an irreducible sub-representation of $\pi|_{K_0}$. Then,*

(1).[3] *The space*

$$\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma,\ \pi)$$

has an eigenvector for the action of the Hecke algebra $\mathscr{H}(K_0, \sigma)$.

(2). *Let λ be an eigenvalue of T in* (1)*. Assume further that:*

$$\lambda \neq \begin{cases} -\chi_1'(-1), & \text{if } \chi_\sigma = \chi_\sigma^s = \chi_1 \otimes \chi_2, \\ 0, & \text{otherwise.} \end{cases}$$

We set a character ε of B such that $\varepsilon|_{H_0} = \chi_\sigma^s$, and

$$\varepsilon(\alpha) = \begin{cases} \lambda + \chi_1'(-1), & \text{if } \chi_\sigma = \chi_\sigma^s, \\ \lambda, & \text{otherwise.} \end{cases}$$

Then, we have the following,

(*a*). *The space in* (1) *is one-dimensional.*

(*b*). *If χ_σ does not factor through the determinant, or $\lambda \neq 1 - \chi_1'(-1)$, then we have*

$$\pi \cong \mathrm{ind}_B^G\, \varepsilon.$$

(*c*). *If χ_σ factors through the determinant, i.e., $\chi_\sigma = \eta \circ \det$ for some character of k_E^1, and $\lambda = 1 - \chi_1'(-1)$. Then*

$$\pi \cong \begin{cases} \eta \circ \det, & \text{if } \dim \sigma = 1, \\ \eta \circ \det \otimes Sp, & \text{otherwise.} \end{cases}$$

Here, Sp is the Steinberg representation of G, defined as $\mathrm{ind}_B^G 1/1$.

[3] This is proved in [Abd11] under the assumption that π is admissible.

In view of the Theorem above, we can now give the definition of supersingular representations. Let T_σ be the following refined Hecke operator:

$$T_\sigma = \begin{cases} T + \chi_1'(-1), & \text{if } \chi_\sigma = \chi_\sigma^s, \\ T, & \text{otherwise.} \end{cases}$$

Definition 1.12. *An irreducible smooth representation π of G is called supersingular if it is a quotient of $\mathrm{ind}_{K_0}^G \sigma/(T_\sigma)$ for some irreducible smooth representation σ of K_0.*

As a by-product of the argument of (1) of Theorem 1.11, we have the following corollary, whose crucial role will become clear soon. Note in this corollary $\tilde{E}$ is not necessarily to be algebraically closed.

Corollary 1.13. *The submodule of $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma, \pi)$ over $\mathscr{H}(K_0, \sigma)$, which is generated by a non-zero G-morphism, is of finite dimension.*

We remark that in Abdellatif's thesis [Abd11], a major part (say (b) and (c) of (2)) of Theorem 1.11 is presented as a conjecture, and assuming the conjecture (and a completely parallel conjecture for the group K_1) she has proved equivalence of supersingular representations and supercuspidal representations for G.

We end this part by recording the following Proposition, which is of independent interest and its argument depends on what we already have described.

Proposition 1.14. [4] *Any non-zero subrepresentation of $\mathrm{ind}_{K_0}^G \sigma$ is non-admissible and reducible. Hence, it is always of infinite length.*

1.4.4 Canonical diagrams and finite presentation

This is the content of chapter 5. For the group $GL_2(\mathbf{Q}_p)$, it is a result of Barthel-Livné and Breuil that all the irreducible smooth representations are finitely presented. However, it seems that is another result only reasonable for $GL_2(\mathbf{Q}_p)$; in fact, the recent work of Hu [Hu12] and Schraen [Sch12] on $GL_2(F)$ has verified that supersingular representations are not finitely presented, when F is either a non-archimedean local field of positive characteristic or a quadratic extension of $\mathbf{Q}_p$. Motivated and following closely the canonical diagrams on GL_2, due to Y. Hu ([Hu12]), chapter 5 is intended to explore similar things for the group G. At the beginning of chapter 5,

[4]It seems that such result is well-known to experts for some time, at least for the case of $GL_2(F)$, but we so far have not found a clear statement of that in literatures.

we have a more detailed description of the underlying motivation and the strategy of Hu and Schraen.

So far, our results are still a little scattered. We will describe what we have proved and what is still expected. To do this, we recall some further notations.

Let σ be an irreducible smooth representation of K_0, and let π be a smooth G-quotient of $\mathrm{ind}_{K_0}^G\sigma$. In 3.7, $R_n^+(\sigma)$ $(n \geq 0)$ is defined as the subspace of $\mathrm{ind}_{K_0}^G\sigma$ which consists of functions supported in $K_0\alpha^n I$, and it is I-stable. One has similar notation $R_n^-(\sigma)$ for $n \geq 0$, which consists of functions supported in $K_0\alpha^{-(n+1)}I$. In terms of the tree of G, there is then a natural I-decomposition of $\mathrm{ind}_{K_0}^G\sigma$:

$$\mathrm{ind}_{K_0}^G\sigma = I^+(\sigma) \oplus I^-(\sigma),$$

where $I^+(\sigma)$ (resp. $I^-(\sigma)$) is $\oplus_{n\geq 0}R_n^+(\sigma)$ (resp. $\oplus_{n\geq 0}R_n^-(\sigma)$). Denote by $I^+(\sigma,\pi)$ (resp. $I^{-1}(\sigma,\pi)$) the image of $I^+(\sigma)$ (resp. $I^-(\sigma)$) in π.

The preliminary results in 5.1 are mainly summarized in Proposition 5.3; besides other things it shows that $I^+(\sigma,\pi) \cap I^-(\sigma,\pi)$ is always non-zero if π is irreducible. We note that the argument of Proposition 5.3 relies crucially on results in previous chapters.

Now we focus on the main results that have been proved. The following two results summarize the contents of 5.2 and 5.3.

Proposition 1.15. *Let π be an irreducible smooth representation of G which is a G-quotient of $\mathrm{ind}_{K_0}^G\sigma$. Let $R(\sigma,\pi)$ be the corresponding kernel. Then the following* (2) *implies* (1) :

(1). $I^+(\sigma,\pi) \cap I^-(\sigma,\pi)$ *is of finite dimension*;

(2). $R(\sigma,\pi)$ *is of finite type, as an $\tilde{E}[G]$-module.*

We indeed expect (1) implies (2) too, but there is some difficulty we have not yet conquered.

From the Hecke operator formula T in chapter 3, for any smooth representation π, we define an endomorphism of π^{N_0} as follows.

Definition 1.16. *For any $v \in \pi^{N_0}$, Sv is defined as*

$$Sv = \textstyle\sum_{u\in N_0/N_2} u\alpha^{-1}\cdot v.$$

This endomorphism S has some nice properties (Lemma 5.12), for example, it preserves I_1-invariants of π. Then we may state the next general result we have arrived at,

Lemma 1.17. *Suppose π is a supersingular representation and a G-quotient of $\mathrm{ind}_{K_0}^G \sigma$. If $0 \neq v \in I^+(\sigma, \pi)$ is fixed by N_0, then there is a polynomial P of degree ≥ 1, such that:*

$$P(S)v = 0.$$

In 5.5, we explore the N_0-invariants of the space $R_{k-1}^+(\sigma) \oplus R_k^+(\sigma) \oplus R_{k+1}^+(\sigma)/T(R_k^+(\sigma))$ for $k \geq 1$ and the outcome is mainly the following partial result, see Remark 5.33 for how it would be expected to be interesting. Denote by C_{N_0} the center of N_0, which is also a pro-p group.

Proposition 1.18. *When $F = \mathbf{Q}_p$, σ is a character of K_0, the dimension of N_0-invariants of $R_{k-1}^+ \oplus R_k^+ \oplus R_{k+1}^+/T_\sigma(R_k^+)$ is at least $p(p-1)$.*

We also expect the Proposition to hold for any irreducible smooth representation of K_0, but we are currently not able to verify it due to some technical reason.

1.4.5 Appendix

In Appendix A, we follow [Paš04] to establish that for the group G, the category of diagrams and that of G-equivariant coefficient systems are equivalent. This part is a bit of formal, and the details are essentially the same as that in [Paš04].

In Appendix B, following [ST02], we give a sufficient condition for the topologically irreducibility of p-adic principal series of $U(1,1)(\mathbf{Q}_{p^2}/\mathbf{Q}_p)$.

2 The pro-p Iwahori-Hecke algebra and its simple modules

In this chapter, we describe the basic structures of the pro-p Iwahori-Hecke algebra $\mathscr{H}_{I_1}$ and its component Iwahori-Hecke algebras $\mathscr{H}(I,\chi)$ of G, and classify their simple modules up to isomorphism. After that, we compute the right action of the pro-p Iwahori–Hecke algebra $\mathscr{H}_{I_1}$ on the I_1-invariants of the principal series. As a result, we define a simple module of $\mathscr{H}_{I_1}$ as supersingular if it is not isomorphic to a sub-quotient of the I_1-invariants of any principal series.

The structure of the pro-p Iwahori-Hecke algebra $\mathscr{H}_{I_1}$ of a connected reductive group (actually the group of its F-points) are relatively well-understood now, mainly from the work of Ollivier, Vignéras. Very briefly speaking, there exists a Bernstein-type basis in $\mathscr{H}_{I_1}$, and the sub-algebra generated by a such basis is commutative and contains the center of $\mathscr{H}_{I_1}$, over which $\mathscr{H}_{I_1}$ is finitely generated. The functor from the category of smooth representations of G to that of modules over $\mathscr{H}_{I_1}$, which sends a smooth representation π to its I_1-invariant π^{I_1}, is expected to play a significant role in the mod-p representation theory. However, also only for very few cases, say $GL_2(\mathbf{Q}_p)$ and $SL_2(\mathbf{Q}_p)$, the full content of this functor is understood thoroughly.

2.1 The structure of $\mathscr{H}(I,\chi)$

Let $\chi=\chi_1\otimes\chi_2$ be a character of I, and $\mathscr{H}(I,\chi)$ be the endomorphism algebra $\mathrm{End}_G\mathrm{ind}_I^G\chi$. From Frobenius reciprocity $\mathscr{H}(I,\chi)\cong(\mathrm{ind}_I^G\chi)^{I,\chi}$, where by definition of isotypic, the latter is the subspace of $\mathrm{ind}_I^G\chi$ consisting of functions φ on G satisfying $\varphi(i_1gi_2)=\chi(i_1i_2)\varphi(g)$ for all $i_1,i_2\in I$ and $g\in G$. From the double coset decomposition of G with respect to I, we deal with the structure of $(\mathrm{ind}_I^G\chi)^{I,\chi}$ in the following:

Lemma 2.1. (1) *Suppose that $\chi=\chi^s$. Then the $\tilde{E}$-space $(\mathrm{ind}_I^G\chi)^{I,\chi}$ has a basis $\{\varphi_{2n,2n+1},\varphi_{2n,2n-1},n\in\mathbb{Z}\}$, where $\varphi_{2n,2n+1}$ (resp. $\varphi_{2n,2n-1}$) is the function supported on $I\alpha^{-n}I$ (resp. $I\beta\alpha^{-n}I$), and is equal to 1 on α^{-n} (resp. $\beta\alpha^{-n}$).*

(2) *Suppose that $\chi\neq\chi^s$. Then a basis for the space $(\mathrm{ind}_I^G\chi)^{I,\chi}$ is $\{\varphi_{2n,2n+1},n\in\mathbb{Z}\}$, where the functions $\varphi_{2n,2n+1}$ are as described in (1).*

Proof. Firstly, we note that the support of a function in $(\mathrm{ind}_I^G\chi)^{I,\chi}$ is a finite union of double cosets IwI, for some $w\in G$ and the restriction of the function to a single coset IwI is determined by its value at w.

Let φ be a function in $(\mathrm{ind}_I^G\chi)^{I,\chi}$ whose support is contained in $I\alpha^{-n}I$. Suppose $\varphi(\alpha^{-n})\neq 0$. Then, for any $i_1,i_2\in I$ satisfying $i_1\alpha^{-n}i_2=\alpha^{-n}$, $\chi(i_1i_2)$ must be equal to 1. This is always true under the condition $i_1^{-1}=\alpha^{-n}i_2\alpha^n$.

For a function φ' in $(\mathrm{ind}_I^G\chi)^{I,\chi}$, whose support is contained in $I\beta\alpha^{-n}I$, suppose $\varphi'(\beta\alpha^{-n})\neq 0$. Then, for any $i_1,i_2\in I$ satisfying $i_1\beta\alpha^{-n}i_2=\beta\alpha^{-n}$, $\chi(i_1i_2)$ must be equal to 1. In other words, $\chi(i_1i_2)=1$ holds whenever $i_1^{-1}=\beta\alpha^{-n}i_2\alpha^n\beta$ is satisfied. But this is to say $\chi=\chi^s$.

The Lemma is shown. □

In the situation $\chi=\chi^s$, let $T_{2n,2n+1},T_{2n,2n-1}$ be the operators in $\mathscr{H}(I,\chi)$ which correspond the functions $\varphi_{2n,2n+1},\varphi_{2n,2n-1}$ respectively. Then, (1) of Lemma 2.1 says these operators together form a basis for the space $\mathscr{H}(I,\chi)$.

We can also form the operators $T_{2n,2n+1}$ for any integer n, in the case that $\chi\neq\chi^s$, which is determined by sending $\varphi_{0,1}$ to $\varphi_{2n,2n+1}$. Similarly, (2) of Lemma 2.1 tells they together form a basis for $\mathscr{H}(I,\chi)$.

Lemma 2.2. 1. $T_{2n,2n+1}=T_{0,-1}T_{2n,2n-1}$*, for* $n\geq 1$.

2. $T_{2n,2n-1}=T_{2,1}T_{2n-2,2n-1}$*, for* $n\geq 1$.
3. $T_{-2n,-2n-1}=T_{0,-1}T_{-2n,-2n+1}$*, for* $n\geq 0$.
4. $T_{-2n-2,-2n-1}=T_{2,1}T_{-2n,-2n-1}$*, for* $n\geq 0$.

Proof. We list the formula for computing the four kinds of operators at the function $\varphi_{0,1}$:

$$
\begin{aligned}
T_{2n,2n+1}(\varphi_{0,1}) &= \sum_{i\in N_1'/N_{2n+1}'} i\alpha^n\cdot\varphi_{0,1},\ n\geq 0,\\
T_{-2n,-2n+1}(\varphi_{0,1}) &= \sum_{i\in N_0/N_{2n}} i\alpha^{-n}\cdot\varphi_{0,1},\ n\geq 0,\\
T_{-2n,-2n-1}(\varphi_{0,1}) &= \sum_{i\in N_0/N_{2n+1}} i\alpha^{-n}\beta\cdot\varphi_{0,1},\ n\geq 0,\\
T_{2n,2n-1}(\varphi_{0,1}) &= \sum_{i\in N_1'/N_{2n}'} i\alpha^{n}\beta\cdot\varphi_{0,1},\ n\geq 1.
\end{aligned}
$$

All these result follows from (11) of [BL94] directly. Then one can check the relations in the Lemma hold without difficulty. We do the first one as an example. We begin with the right side product, say, for $n\geq 1$,

$$
\begin{aligned}
T_{0,-1}\cdot T_{2n,2n-1}(\varphi_{0,1}) &= T_{0,-1}\left(\sum_{i\in N_1'/N_{2n}'} i\alpha^n\beta\cdot\varphi_{0,1}\right)\\
&= \sum_{i\in N_1'/N_{2n}'} i\alpha^n\beta\sum_{j\in N_0/N_1} j\beta\cdot\varphi_{0,1}\\
&= \sum_{i\in N_1'/N_{2n}'} i\sum_{j\in N_0/N_1} \alpha^n\beta j\beta\alpha^{-n}\cdot\alpha^n\varphi_{0,1}\\
&= \sum_{i\in N_1'/N_{2n}'} i\sum_{j'\in N_{2n}'/N_{2n+1}'} j'\alpha^n\varphi_{0,1}\\
&= \sum_{i\in N_1'/N_{2n+1}'} i'\alpha^n\varphi_{0,1}\\
&= T_{2n,2n+1}(\varphi_{0,1}),
\end{aligned}
$$

we are done. ∎

It is immediate from Lemma 2.2 that

Corollary 2.3. 1. $T_{2n,2n+1}=(T_{0,-1}T_{2,1})^n$, *for* $n\geq 1$.

2. $T_{2n,2n-1}=T_{2,1}(T_{0,-1}T_{2,1})^{n-1}$, *for* $n\geq 1$.
3. $T_{-2n,-2n-1}=T_{0,-1}(T_{2,1}T_{0,-1})^{n}$, *for* $n\geq 0$.
4. $T_{-2n-2,-2n-1}=(T_{2,1}T_{0,-1})^{n+1}$, *for* $n\geq 0$.

Remark 2.4. *Any element in* $\mathscr{H}(I,\chi)$ *is a unique linear combination of monomials of the forms in Corollary 2.3, i.e., the set*

$$\{(T_{0,-1}T_{2,1})^n, T_{2,1}(T_{0,-1}T_{2,1})^n, T_{0,-1}(T_{2,1}T_{0,-1})^n, (T_{2,1}T_{0,-1})^{n+1}\}_{n\geq 0}$$

consists of a basis of $\mathscr{H}(I,\chi)$; in particular, $\mathscr{H}(I,\chi)$ is generated by the two operators $T_{0,-1}$ and $T_{2,1}$. As a result, it makes sense to define the degree of an element in $\mathscr{H}(I,\chi)$ as the highest degree of the terms in its unique expression. We emphasize that the degree of an element defined here should not be viewed as the 'degree' of the element as a polynomial in $T_{0,-1}$ and $T_{2,1}$.

To unify many calculations in this section, we record a simple fact in the following lemma

Lemma 2.5. (1) *If $\beta\alpha^k i\alpha^l \in I\beta\alpha^m I$ for some $i\in N_1'$ or N_0, then $k+l=m$;*
(2) *If $\alpha^k i\alpha^l \in I\alpha^m I$ for some $i\in N_1'$ or N_0, then $k+l=m$.*

Proof. We check (1) in detail where (2) follows in the same way. For (1), we only need to consider two cases: $(i) : i\in N_1', k<0, l>0$; $(ii) : i\in N_0, k\geq 0, l<0$. For example, for $i\in N_1', k\geq 0$, or $l\leq 0$, (1) obviously holds: $\beta\alpha^k i\alpha^l = (\beta\alpha^k i\alpha^{-k}\beta)\beta\alpha^k\alpha^l \in I\beta\alpha^{k+l}$.

The assumption means there are elements $i_1, i_2\in I$ such that

$$\alpha^{-k}\beta i_1\beta\alpha^m i_2\alpha^{-l}\in I.$$

In the case (i), from the decomposition $I = N_1'\cdot(I\cap B) = (I\cap B')\cdot N_0$, one could assume further that $i_2\in N_1'$ and $i_1\in N_0$; as a result, the product of elements above is a lower triangular matrix in I and one concludes $k+l=m$ immediately from the fact that the diagonal entries of I lie in the units of E.

The same trick applies to case (ii). We are done. $\square$

Proposition 2.6. *Let $\chi=\chi_1\otimes\chi_2$ be a character of I satisfying $\chi=\chi^s$, and let χ_1' be the character of k_E^1 given by Remark 1.4.*

(1). *Suppose χ does not factor through the determinant. Then,*

$$\mathscr{H}(I,\chi)\cong \tilde{E}[T_{0,-1},T_{2,1}]/(T_{0,-1}^2, T_{2,1}^2+\chi_1'(-1)T_{2,1}).$$

(2). *Suppose χ factors through the determinant. Then as an $\tilde{E}$-algebra,*

$$\mathscr{H}(I,\chi)\cong \tilde{E}[T_{0,-1},T_{2,1}]/(T_{0,-1}^2+\chi_1'(-1)T_{0,-1}, T_{2,1}^2+\chi_1'(-1)T_{2,1}).$$

Proof. We note that the remark before Lemma 2.2 and Corollary 2.3 tells that $T_{0,-1}, T_{2,1}$ generate $\mathscr{H}(I,\chi)$. We compute first $T_{0,-1}^2(\varphi_{0,1})$.

By definition,

$$T_{0,-1}^2(\varphi_{0,1}) = T_{0,-1}(\textstyle\sum_{i\in N_0/N_1} i\beta\varphi_{0,1}) = \sum_{i\in N_0/N_1} i\beta\cdot\varphi_{0,-1}.$$

In general, the function above is supported on a finite union of double cosets with a form of $I\alpha^k I$ or $I\beta\alpha^k I$. We recall $\varphi_{0,-1}$ is supported on $I\beta I$. If $\alpha^k i\beta \in I\beta I$ for some $k \in \mathbb{Z}$ and some $i \in N_0$, then $k = 0$ from Lemma 2.5; similarly, if $\beta\alpha^{k'} i\beta \in I\beta I$ for some $k' \in \mathbb{Z}$ and some $i \in N_0$, then clearly $i \in N_0 \setminus N_1$, and then one must have that $k' = 0$: if $k' < 0$, then $\beta\alpha^{k'} i\beta = (\beta\alpha^{k'} i\alpha^{-k'}\beta)\beta\alpha^{k'}\beta \in I\alpha^{-k'}I$; if $k' > 0$, using Lemma 1.2, one see $\beta\alpha^{k'} i\beta = \alpha^{-k'} i_1\beta i_2 \in I\beta\alpha^{k'} I$, for some $i_1 \in N_0$, $i_2 \in I \cap B$. In summary, we have shown the support of $T^2_{0,-1}(\varphi_{0,1})$ is contained in $I \cup I\beta I$.

We see first

$$\begin{aligned}\sum_{i\in N_0/N_1} i\beta \cdot \varphi_{0,-1}(Id) &= \sum_{i\in N_0/N_1} \varphi_{0,-1}(i\beta)\\ &= \sum_{i\in N_0/N_1} 1 = 0.\end{aligned}$$

Hence, $T^2_{0,-1}$ differs from $T_{0,-1}$ by a constant factor.

Next,

$$\begin{aligned}\sum_{i\in (N_0\setminus N_1)/N_1} i\beta \cdot \varphi_{0,-1}(\beta) &= \sum_{i\in (N_0\setminus N_1)/N_1} \varphi_{0,-1}(\beta i\beta)\\ &= \sum_{i\in n(L_2^*)} \varphi_{0,-1}(\beta i\beta).\end{aligned}$$

Here $L_2^* = L_2 \ \setminus \{(0,0)\}$.

By writing i as $n(x,y)$, for $(x,y) \in L_2^*$, an application of Lemma 1.2 implies $\beta i\beta = uh_i u'\beta$ for some $u \in N_0$, $u'_0 \in N'_0$ and $h_i = \mathrm{diag}(\bar{y}^{-1}, -\bar{y}y^{-1}, y)$, hence we are led to

$$\sum_{i\in (N_0\setminus N_1)/N_1} i\beta \cdot \varphi_{0,-1}(\beta) = \sum_{(x,y)\in L_2^*} \chi_1(\bar{y}^{-1})\chi_2(-\bar{y}y^{-1}). \tag{4}$$

We compute at first the part $\sum_1$ of (4) in which x is zero. In this case, $y = -\bar{y}$, hence

$$\textstyle\sum_1 = \sum_{y\in L_1^*} \chi_1(\bar{y}^{-1}) = \sum_{y\in L_1^*} \chi_1(y).$$

As χ is trivial on I_1, we could identify χ_1 and χ_2 with respectively characters of $k_E^\times$ and k_E^1. Then from Remark 1.4, the assumption on χ gives us a unique character χ'_1 of k_E^1, such that $\chi_1(y) = \chi'_1(y\bar{y}^{-1})$. Then,

$$\textstyle\sum_1 = \sum_{y=-\bar{y}, y\in k_E^\times} \chi_1(y) = \sum_{y=-\bar{y}, y\in k_E^\times} \chi'_1(y\bar{y}^{-1}) = -\chi'_1(-1).$$

Denote $\chi_2(\chi_1')^{-1}$ by χ', then χ' is a character of k_E^1 and it is trivial if and only if χ factors through the determinant (Remark 1.4). The remaining part, i.e., the sum over terms in (4) for which x is non-zero, can be written as

$$\sum\nolimits_2 = \chi_1'(-1) \sum\nolimits_{x\neq 0} \chi'(-\bar{y}y^{-1}),$$

in which the sum

$$\begin{aligned}\sum_{x\neq 0} \chi'(-\bar{y}y^{-1}) &= \sum_{t\in k_F^\times} \sum_{x\bar{x}=-t} \sum_{y+\bar{y}=t} \chi'(-\bar{y}y^{-1}) \\ &= \sum_{t\in k_F^\times} \sum_{y+\bar{y}=t} \chi'(-\bar{y}y^{-1}) \\ &= \sum_{\mathrm{tr}(y)\neq 0} \chi'(-\bar{y}y^{-1}).\end{aligned}$$

Now, $$\begin{aligned}\sum_{\mathrm{tr}(y)\neq 0} \chi'(-\bar{y}y^{-1}) &= \chi'(-1) \sum_{y\in k_E^\times} \chi'(\bar{y}y^{-1}) - \sum_{\mathrm{tr}(y)=0,\ y\in k_E^\times} \chi'(-\bar{y}y^{-1}) \\ &= \chi'(-1) \sum_{l\in k_E^1} (q-1)\chi'(l) - (q-1) \\ &= 1 - \chi'(-1) \sum_{l\in k_E^1} \chi'(l).\end{aligned}$$

Now it is well-known that the last sum above is 1 if χ' is non-trivial; otherwise it is $1-(q+1)=0$, where $q+1$ is the order of k_E^1. We are done for the argument for the quadratic relations of $T_{0,-1}$.

We now show $T_{2,1}\cdot T_{2,1} = -\chi_1'(-1)T_{2,1}$. By definition,

$$T_{2,1}\cdot T_{2,1}(\varphi_{0,1}) = \sum\nolimits_{j\in N_1'/N_2'} j\alpha\beta\cdot\ \varphi_{2,1}.$$

We note that $\varphi_{2,1}$ is supported on $I\beta\alpha^{-1}I$, and the above function is supported on double cosets of the form $I\alpha^k I$ or $I\beta\alpha^k I$. That $\alpha^k j\alpha\beta \in I\beta\alpha^{-1}I$ for some $k\in\mathbb{Z}$ and some $j\in N_1'$, forces $k=0$, from Lemma 2.5; similarly, if $\beta\alpha^{k'}j\alpha\beta \in I\beta\alpha^{-1}I$, for some $k'\in\mathbb{Z}$ and some $j\in N_1'$, then clearly $j\in N_1'\setminus N_2'$, and an application of Lemma 1.2 reduces the situation to Lemma 2.5, which forces $k'=-1$. Therefore, we have shown the support of $T_{2,1}^2(\varphi_{0,1})$ is contained in $I\cup I\beta\alpha^{-1}I$. Furthermore, we have

$$\sum\nolimits_{j\in N_1'/N_2'} j\alpha\beta\cdot\ \varphi_{2,1}(Id) = \sum\nolimits_{j\in N_1'/N_2'} \varphi_{2,1}(j\alpha\beta) = 0$$

and,

$$\sum_{j\in N_1'/N_2'} j\alpha\beta\cdot\ \varphi_{2,1}(\beta\alpha^{-1}) = \sum_{j\in(N_1'\setminus N_2')/N_2'} \varphi_{2,1}(\beta\alpha^{-1}j\alpha\beta).$$

An application of Lemma 1.2 to $\beta\alpha^{-1}j\alpha\beta$ for $j\in(N_1'\setminus N_2')/N_2'$ reduces the above sum into

$$\sum_{y=-\bar{y},y\in k_E^\times} \chi_1(y)$$

which is just $-\chi_1'(-1)$ as we have determined before. This confirms the quadratic relation for $T_{2,1}$.

A few words is needed to complete the proof of the Proposition. We will do (2) for example. There is a natural homomorphism κ from the polynomial ring $\tilde{E}[T_{0,-1},T_{2,1}]$ to $\mathscr{H}(I,\chi)$, which is surjective by Corollary 2.3. From the quadratic relations of $T_{0,1}$ and $T_{2,1}$ we have just proved, the κ will factor as

$$\kappa:\ \tilde{E}[T_{0,-1},T_{2,1}]/(T_{0,-1}^2+\chi_1'(-1)T_{0,-1},T_{2,1}^2+\chi_1'(-1)T_{2,1})\to\mathscr{H}(I,\chi).$$

For any element f in $\tilde{E}[T_{0,-1},T_{2,1}]/(T_{0,-1}^2+\chi_1'(-1)T_{0,-1},T_{2,1}^2+\chi_1'(-1)T_{2,1})$, one could choose its representative in $\tilde{E}[T_{0,-1},T_{2,1}]$ as a linear combination of monomials of the form in Corollary 2.3. By Remark 2.4, it is then clear that f vanishes if $\kappa(f)=0$. We are done. $\square$

Remark 2.7. *In the argument of the last proposition, we have determined the value of the sum*

$$\sum_{y=-\bar{y},y\in k_E^\times} \chi_1(y)$$

and the sum

$$\sum_{(x,y)\in L_2^*} \chi_1(\bar{y}^{-1})\chi_2(-\bar{y}y^{-1})$$

when $\chi=\chi_1\otimes\chi_2$ satisfies $\chi=\chi^s$. Later on, we will also compute their values in the remaining case.

We turn to the case that $\chi\neq\chi^s$.

Lemma 2.8. (1). $T_{2,3}\cdot T_{-2,-1}=0$, $T_{-2,-1}\cdot T_{2,3}=0$.
(2). *For $n\geq 0$, $T_{2n,2n+1}=(T_{2,3})^n$, $T_{-2n,-2n+1}=(T_{-2,-1})^n$.*

Proof. We prove first that $T_{2,3}\cdot T_{-2,-1}=0$. By definition,

$$T_{2,3}\cdot T_{-2,-1}(\varphi_{0,1})=\sum_{i\in N_0/N_2} i\alpha^{-1}\varphi_{2,3}.$$

We remind the reader that this function will be supported on a union of double cosets with a form of $I\alpha^kI$, for some integers k. Recall $\varphi_{2,3}$ is supported on $I\alpha^{-1}I$. If $\alpha^k i\alpha^{-1}\in I\alpha^{-1}I$, for some $k\in\mathbb{Z}$ and $i\in N_0$, then $k=0$, from Lemma 2.5. We compute

$$\sum_{i\in N_0/N_2} i\alpha^{-1}\varphi_{2,3}(Id) = 0,$$

as required.

Similarly,

$$T_{-2,-1}\cdot T_{2,3}(\varphi_{0,1}) = \sum_{i\in N_1'/N_3'} i\alpha\cdot\varphi_{-2,-1}.$$

Also, the above function is supported on a union of double cosets with a form of $I\alpha^k I$, for some integers k. Note that $\varphi_{-2,-1}$ is supported on $I\alpha I$. If $\alpha^k i\alpha \in I\alpha I$, for some $k\in\mathbb{Z}$ and some $i\in N_1'$, then $k=0$, from Lemma 2.5. We compute

$$\sum_{i\in N_1'/N_3'} i\alpha\cdot\varphi_{-2,-1}(Id) = 0,$$

also as required. We are done for (1).

The formulae in the proof of Lemma 2.2 hold if the operators makes sense. Then (2) follows from those formulae by induction. □

We can state the following Proposition:

Proposition 2.9. *For a character χ of I which satisfies $\chi\neq\chi^s$, we have an isomorphism of $\tilde{E}$-algebra:*

$$\mathscr{H}(I,\chi)\cong\tilde{E}[T_{2,3},T_{-2,-1}]/(T_{2,3}\cdot T_{-2,-1},\ T_{-2,-1}\cdot T_{2,3}).$$

Proof. This follows from Lemma 2.8 and the remark before Lemma 2.2. □

Remark 2.10. *In the following sections, sometimes we will use the notations $T^\chi_{2,3}$ and $T^\chi_{-2,-1}$ for $T_{2,3}$ and $T_{-2,-1}$ to indicate that they are in $\mathscr{H}(I,\chi)$, not in $\mathscr{H}(I,\chi^s)$, when $\chi\neq\chi^s$.*

2.2 Simple modules over Iwahori-Hecke algebras

After determining the structure of $\mathscr{H}(I,\chi)$, we turn to the simple modules over them. We begin to investigate the center $\mathscr{C}_\chi$ of $\mathscr{H}(I,\chi)$. We have shown $\mathscr{H}(I,\chi)$ is commutative if $\chi\neq\chi^s$. Hence, only the case $\chi=\chi^s$ is need to be considered:

Lemma 2.11. (1). *Let χ be a character of I which factors through determinant, then $\mathscr{C}_\chi\cong\tilde{E}[c]$, where c is the operator*

$$1+\chi_1'(-1)\cdot T_{0,-1}+\chi_1'(-1)\cdot T_{2,1}+T_{0,-1}\cdot T_{2,1}+T_{2,1}\cdot T_{0,-1}.$$

(2). *Let χ be a character of I which is fixed by s, but it does not factor through determinant, then $\mathscr{C}_\chi\cong\tilde{E}[c']$, where c' is the operator*

$$\chi_1'(-1)\cdot T_{0,-1}+T_{0,-1}\cdot T_{2,1}+T_{2,1}\cdot T_{0,-1}.$$

Proof. It can easily be checked that the operators c and c' are in the center.

We will prove (1) in detail. Suppose χ is a character of I which factors through the determinant, and we see $\tau: \mathscr{H}(I,\chi) \cong \mathscr{H}(I,1)$. This isomorphism comes from the isomorphism $\iota: \mathrm{ind}_I^G\chi \cong \chi' \otimes \mathrm{ind}_I^G 1$ of representations, where the χ' on the right hand is an extension of χ to G. We write ι explicitly on the basis elements $g\varphi_{0,1}$:

$$\iota: \ g\varphi_{0,1} \mapsto \chi'(g)1_{Ig^{-1}},$$

from which the basis $\{T_{2n,2n+1}, T_{2n,2n-1}, n \in \mathbb{Z}\}$ correspond via τ to the following:

$$T_{2n,2n+1} \mapsto T_{2n,2n+1},\ T_{2n,2n-1} \mapsto \chi'(\beta)T_{2n,2n-1},$$

but, $\chi'(\beta) = \chi_1'(-1)$ as one can check.

For (1), we then reduce to the case $\mathscr{H}(I,1) = \mathscr{H}_I$. It results from the following Lemma:

Lemma 2.12. *Denote by $\mathscr{C}$ the center of $\mathscr{H}_I$.*

(1) *Each non-zero element of $\mathscr{C}$ is of even degree.*

(2) *For a non-zero element in $\mathscr{C}$ with degree bigger than 0, its coefficients of the two terms of highest degree are the same.*

Proof. We note that the statements make sense from Remark 2.4. Both can be checked directly. □

Given a non-zero $t \in \mathscr{C}$, by (2) of Lemma 2.12, one could find some non-zero $a \in \tilde{E}$ and a non-negative integer n such that $t - ac^n (\in \mathscr{C})$ has smaller degree than t. If it is zero, we stop. Otherwise, we repeat the former process for a finite times, which finally leads to zero, i.e., $t \in \tilde{E}[c]$.

We have proved (1) for $\mathscr{H}_I$. Via the isomorphism H, we have indeed shown (1) in general.

For (2), one firstly shows an analogue of Lemma 2.12, then the result follows from that as above. □

Remark 2.13. *From Lemma 2.11, $\mathscr{H}(I,\chi)$ is finite over $\mathscr{C}_\chi$ as an algebra. In fact, $\mathscr{H}(I,\chi)$ admits bigger commutative algebras, for example, $\mathscr{C}_\chi[T_{2,1}]$, over which $\mathscr{H}(I,\chi)$ is of rank two, with the basis $\{1,\ T_{0,-1}\}$. If $\tilde{E}$ is algebraically closed, the simple modules of $\mathscr{H}(I,\chi)$ are at most two dimensional. Later on, we assume $\tilde{E}$ is algebraically closed.*

Proposition 2.14. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of I which is fixed by s. The characters of $\mathscr{H}(I,\chi)$ are the following:*

If χ factors through the determinant,

(1). $T_{0,-1} \mapsto 0,\ T_{2,1} \mapsto 0$;
(2). $T_{0,-1} \mapsto 0,\ T_{2,1} \mapsto -\chi_1'(-1)$;
(3). $T_{0,-1} \mapsto -\chi_1'(-1),\ T_{2,1} \mapsto 0$;
(4). $T_{0,-1} \mapsto -\chi_1'(-1),\ T_{2,1} \mapsto -\chi_1'(-1)$.

If χ does not factor through the determinant,
(1′). $T_{0,-1} \mapsto 0,\ T_{2,1} \mapsto 0$;
(2′). $T_{0,-1} \mapsto 0,\ T_{2,1} \mapsto -\chi_1'(-1)$.

Here we understand a character of an algebra as a morphism from itself to the coefficient field, hence in the above we only specify the values of the generators.

Let $\chi = \chi_1 \otimes \chi_2$ be a character of I which is fixed by s. Let $\langle v_1, v_2\rangle_{\tilde{E}}$ be a two dimensional $\tilde{E}$-vector space, on which a two dimensional simple right $\mathscr{H}(I,\chi)$-module is defined by:

Definition 2.15. *If χ factors through the determinant, $\lambda \in \tilde{E}, \lambda \neq 0, 1$,*

$$(v_1, v_2)\, T_{0,-1} = (v_1, v_2) \begin{pmatrix} 0 & 0 \\ 1 & -\chi_1'(-1) \end{pmatrix},$$
$$(v_1, v_2)\, T_{2,1} = (v_1, v_2) \begin{pmatrix} -\chi_1'(-1) & \lambda \\ 0 & 0 \end{pmatrix}.$$

The central operator c acts as scalar λ.

If χ does not factor through the determinant, $\lambda \in \tilde{E}, \lambda \neq 0$,

$$(v_1, v_2)\, T_{0,-1} = (v_1, v_2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$(v_1, v_2)\, T_{2,1} = (v_1, v_2) \begin{pmatrix} -\chi_1'(-1) & \lambda \\ 0 & 0 \end{pmatrix}.$$

The central operator c' acts as scalar λ.

Proposition 2.16. *Any two dimensional simple module of $\mathscr{H}(I,\chi)$, on which the central operator c (or c') acts as a scalar λ, is isomorphic to the corresponding one defined in 2.15.*

Proof. We show the first case in detail.

Let M be a two dimensional simple module over $\mathscr{H}(I,\chi)$, on which the central operator c acts as some $\lambda \in \tilde{E}$. From the quadratic relation for $T_{0,-1}$, we can choose a basis $\{v_1,\ v_2\}$ in the underlying space of M, such that :

$$(v_1, v_2)\, T_{0,-1} = (v_1, v_2) \begin{pmatrix} 0 & 0 \\ 1 & -\chi_1'(-1) \end{pmatrix}.$$

Here, it is clear $T_{0,-1}$ can not act as a scalar, otherwise $T_{2,1}$ would act as a scalar too, due to the assumption on the central operation c.

Assume the matrix of $T_{2,1}$ under the above basis is (a_{ij}), then from the condition that c acts as λ and the quadratic relation for $T_{2,1}$, we see immediately:

$$\lambda = 1 + a_{11}\chi_1'(-1) + a_{12},\ \ a_{11} + a_{22} = -\chi_1'(-1),$$
$$a_{12}a_{21} = -a_{11}(a_{11} + \chi_1'(-1)).$$

The simplicity of M implies that $\lambda \neq 0, 1$, from which we see $a_{12} \neq 0$.

We choose another basis, say $\{v_1', v_2'\}$, where

$$v_1' = v_1 + \chi_1'(-1)(1 - a_{12}^{-1}\lambda)v_2,\ \ v_2' = a_{12}^{-1}\lambda v_2,$$

then one see that the matrices of $T_{0,-1}$ and $T_{2,1}$ under the new basis are those stated in Definition 2.15. We are done.

The second case could be shown similarly, and we don't give details here. □

2.3 Second Iwahori-Hecke algebras and their simple modules

Let χ be a character of I which satisfies $\chi \neq \chi^s$. Recall we have shown in Proposition 2.9 that $\mathscr{H}(I,\chi) \cong \tilde{E}[T_{2,3}, T_{-2,-1}]/(T_{2,3} \cdot T_{-2,-1},\ T_{-2,-1} \cdot T_{2,3})$. *In this section, we use the notation $T^{\chi}_{2,3}$ and $T^{\chi}_{-2,-1}$ for $T_{2,3}$ and $T_{-2,-1}$ to avoid confusion.* For any integer n, let ϕ_{n,χ^s} be the function in $\mathrm{ind}_I^G \chi^s$, supported on $I\beta\alpha^n I$, and $\phi_{n,\chi^s}(\beta\alpha^n) = 1$, and $\phi_{n,\chi^s}(i_1 g i_2) = \chi^s(i)\phi_{n,\chi^s}(g)\chi(i_2)$, for any $i_1, i_2 \in I, g \in G$. Then $\{\phi_{n,\chi^s},\ n \in \mathbb{Z}\}$ is a basis of $(\mathrm{ind}_I^G \chi^s)^{I,\chi}$. Via Frobenius reciprocity, we get a basis $\{\psi_{n,\chi^s},\ n \in \mathbb{Z}\}$ for $\mathrm{Hom}_G(\mathrm{ind}_I^G \chi,\ \mathrm{ind}_I^G \chi^s)$, in which ψ_{n,χ^s} is determined by $\psi_{n,\chi^s}(\varphi_\chi) = \phi_{n,\chi^s}$, where φ_χ is the function in $\mathrm{ind}_I^G \chi$, supported on I and taking value 1 at the identity of G.

Proposition 2.17. *The morphisms ψ_{0,χ^s}, ψ_{-1,χ^s} generate* $\mathrm{Hom}_G(\mathrm{ind}_I^G \chi,\ \mathrm{ind}_I^G \chi^s)$, *as an $\mathscr{H}(I,\chi^s)$-$\mathscr{H}(I,\chi)$-bi-module. In more words,*

(1)

$$\text{For } n \geq 0,\ \psi_{n,\chi^s} \circ T^{\chi}_{2,3} = 0,\ \psi_{n,\chi^s} \circ T^{\chi}_{-2,-1} = \psi_{n+1,\chi^s},$$
$$\text{For } n > 0, \psi_{-n,\chi^s} \circ T^{\chi}_{2,3} = \psi_{-(n+1),\chi^s},\ \psi_{-n,\chi^s} \circ T^{\chi}_{-2,-1} = 0.$$

(2)

$$\textit{For } n \geq 0,\ T^{\chi^s}_{-2,-1} \circ \psi_{n,\chi^s} = 0,\ T^{\chi^s}_{2,3} \circ \psi_{n,\chi^s} = \psi_{n+1,\chi^s},$$
$$\textit{For } n > 0,\ T^{\chi^s}_{-2,-1} \circ \psi_{-n,\chi^s} = \psi_{-(n+1),\chi^s},\ T^{\chi^s}_{2,3} \circ \psi_{-n,\chi^s} = 0.$$

Proof. We will verify the formulas in (1) in detail, where those in (2) follow completely in the same way, using the formulas in Lemma 2.2 and the definitions. Note that the first statement follows from the formulas in the Proposition.

By definition, for $n \geq 0$,

$$\psi_{n,\chi^s} \circ T^{\chi}_{2,3}(\varphi_\chi) = \psi_{n,\chi^s}(\textstyle\sum_{i \in N_1'/N_3'} i\alpha \cdot \varphi_\chi) = \sum_{i \in N_1'/N_3'} i\alpha \cdot \phi_{n,\chi^s}.$$

We recall the above function is supported on a union of double cosets with a form of $I\beta\alpha^k I$. Note ϕ_{n,χ^s} is supported on $I\beta\alpha^n I$, and $\beta\alpha^k i\alpha$ lies in $I\beta\alpha^n I$ for some $i \in N_1'/N_3'$ only if $k = n-1$, from Lemma 2.5.

When $n > 0$, we have

$$\begin{aligned}\sum_{i \in N_1'/N_3'} i\alpha \cdot \phi_{n,\chi^s}(\beta\alpha^{n-1}) &= \sum_{i \in N_1'/N_3'} \phi_{n,\chi^s}(\beta\alpha^{n-1} i \alpha^{1-n}\beta\beta\alpha^n) \\ &= \sum_{i \in N_1'/N_3'} 1 = 0.\end{aligned}$$

When $n = 0$, we also have

$$\textstyle\sum_{i \in N_1'/N_3'} \phi_{0,\chi^s}(\beta\alpha^{-1} i\alpha) = \sum_{i \in N_2'/N_3'} \phi_{0,\chi^s}(\beta\alpha^{-1} i\alpha) = 0,$$

where we note that $\beta\alpha^{-1} i\alpha \in I\alpha^{-1} I$ for $i \in N_1' \setminus N_2'$ by applying Lemma 1.2.

We have verified $\psi_n \circ T_{2,3} = 0$, for $n \geq 0$.

Similarly, we have from definitions that, for $n > 0$

$$\psi_{-n,\chi^s} \circ T^{\chi}_{2,3}(\varphi_\chi) = \textstyle\sum_{i \in \in N_1'/N_3'} i\alpha \cdot \phi_{-n,\chi^s}.$$

Also, the above function is supported on a union of double cosets with a form of $I\beta\alpha^k I$, for some integers k. We remind the reader ϕ_{-n,χ^s} is supported on $I\beta\alpha^{-n} I$. If $\beta\alpha^k i\alpha$ lies in $I\beta\alpha^{-n} I$, for some $i \in N_1'/N_3'$, then $k = -(n+1)$, from Lemma 2.5. We proceed to compute

$$\textstyle\sum_{i \in N_1'/N_3'} i\alpha \cdot \phi_{-n,\chi^s}(\beta\alpha^{-(n+1)}) = \sum_{i = Id} \phi_{-n,\chi^s}(\beta\alpha^{-(n+1)} i\alpha) = 1,$$

where we claim that $\beta\alpha^{-(n+1)} i\alpha \notin I\beta\alpha^{-n} I$ for $i \in N_1' \setminus N_3'$.

Recall we are in the case that $n > 0$. Assume there are elements $i_1, i_2 \in I$ such that

$$\alpha^{n+1}\beta i_1 \beta\alpha^{-n} i_2 \alpha^{-1} \in N_1' \setminus N_3'. \tag{5}$$

Using the decomposition $I = (B \cap I) \cdot N_1'$ and multiplying both sides of the above identity by some elements in N_3', one could assume $i_2 \in B \cap I$ and $i_1 \in B' \cap I$, where a contradiction arises from the fact that an upper triangular matrix can not be a non-trivial lower unipotent matrix. Hence the claim. *In the following many calculations, the above trick to check something like* (5) *does not hold will appear frequently and we will cite it to avoid the repeated computations.*

Also, for $n \geq 0$,

$$\psi_{n,\chi^s} \circ T^{\chi}_{-2,-1}(\varphi_\chi) = \textstyle\sum_{i \in N_0/N_2} \, i\alpha^{-1}\phi_{n,\chi^s}.$$

We note ϕ_{n,χ^s} is supported on $I\beta\alpha^n I$. If $\beta\alpha^k i\alpha^{-1}$ lies in $I\beta\alpha^n I$, for some $i \in N_0/N_2$, then $k = n+1$, from Lemma 2.5. We compute

$$\textstyle\sum_{i \in N_0/N_2} \, i\alpha^{-1}\phi_{n,\chi^s}(\beta\alpha^{n+1}) = \sum_{i=Id} \phi_{n,\chi^s}(\beta\alpha^{n+1}i\alpha^{-1}) = 1,$$

where one could check that $\beta\alpha^{n+1}i\alpha^{-1} \notin I\beta\alpha^n I$ for $i \in N_0 \setminus N_2$, via the same process used in the first case.

Lastly, for $n > 0$,

$$\psi_{-n,\chi^s} \circ T^{\chi}_{-2,-1}(\varphi_\chi) = \textstyle\sum_{i \in N_0/N_2} \, i\alpha^{-1}\phi_{-n,\chi^s}.$$

Note ϕ_{-n,χ^s} is supported on $I\beta\alpha^{-n}I$. If $\beta\alpha^k i\alpha^{-1}$ lies in $I\beta\alpha^{-n}I$, for some $i \in N_0/N_2$, then $k = -n+1$, from Lemma 2.5. And we compute

$$\begin{aligned}\sum_{i \in N_0/N_2} i\alpha^{-1}\phi_{-n,\chi^s}(\beta\alpha^{-n+1}) = &\sum_{i \in N_1/N_2} \phi_{-n,\chi^s}(\beta\alpha^{-n+1}i\alpha^{n-1}\beta\beta\alpha^{-n}) \\ &+ \sum_{i \in (N_0 \setminus N_1)/N_2} \phi_{-n,\chi^s}(\beta\alpha^{1-n}i\alpha^{-1}).\end{aligned}$$

In the above, the first sum is clearly zero. We check the second sum is also zero. When $n = 1$, we have $\beta i\alpha^{-1} \in I\alpha^{-1}I$ for $i \in N_0 \setminus N_1$. When $n > 1$, we see $\beta\alpha^{-n+1}i\alpha^{n-1}\beta \in N_1'$ for $i \in N_0 \setminus N_1$ and the cardinality of $(N_0 \setminus N_1)/N_2$ is $(q^3-1)q$. The claim is then verified.

We are done for the proof of (1). □

Remark 2.18. *We have a natural isomorphism between the algebra $\mathscr{H}(I,\chi)$ and $\mathscr{H}(I,\chi^s)$, which is determined by sending $T^{\chi}_{2,3}$ and $T^{\chi}_{-2,-1}$ to $T^{\chi^s}_{-2,-1}$ and $T^{\chi^s}_{2,3}$ respectively. Under this isomorphism, Proposition 2.17 says the bi-module structures of* $\mathrm{Hom}_G(\mathrm{ind}_I^G\chi,\, \mathrm{ind}_I^G\chi^s)$ *coincide.*

Corollary 2.19. *As a right module over $\mathscr{H}(I,\chi)$, we have*

$$\mathrm{Hom}_G(\mathrm{ind}_I^G\chi, \mathrm{ind}_I^G\chi^s) \cong \mathscr{H}(I,\chi)/(T^{\chi}_{2,3}) \oplus \mathscr{H}(I,\chi)/(T^{\chi}_{-2,-1}).$$

Proof. From the descriptions of the space $\mathrm{Hom}(\mathrm{ind}_I^G\chi,\ \mathrm{ind}_I^G\chi^s)$ in Proposition 2.17, we have a natural surjective $\mathscr{H}(I,\chi)$-morphism from $\mathscr{H}(I,\chi)\oplus\mathscr{H}(I,\chi)$ to $\mathrm{Hom}(\mathrm{ind}_I^G\chi,\ \mathrm{ind}_I^G\chi^s)$, and the kernel is given by Proposition 2.17. We are done. □

Remark 2.20. *In contrast to the case of GL_2 ([Vig04]), we point out a corollary from last result that $\mathrm{ind}_I^G\chi$ is* not *G-isomorphic to $\mathrm{ind}_I^G\chi^s$.*

Proposition 2.21.

$$\psi_{n,\chi}\circ\psi_{m,\chi^s}=\begin{cases}0, & \text{if } m\geq 0, n\geq 0,\\ (T^{\chi}_{-2,-1})^{m+l}, & \text{if } m\geq 0,\ n=-l<0,\\ (T^{\chi}_{2,3})^{t+n}, & \text{if } m=-t<0,\ n\geq 0,\\ 0, & \text{if } m<0,\ n<0.\end{cases} \tag{6}$$

Proof. In view of Proposition 2.17, the results are reduced to checking some initial cases.

For $n\geq 0$ and $m\geq 0$, we have, from Proposition 2.17,

$$\psi_{n,\chi}\circ\psi_{m,\chi^s}=\psi_{0,\chi}\circ(T^{\chi^s}_{-2,-1})^n\circ(T^{\chi^s}_{2,3})^m\circ\psi_{0,\chi^s};$$

hence we only need to treat the case $n=m=0$. By definition, we have

$$F_{0,0}:=\psi_{0,\chi}\circ\psi_{0,\chi^s}(\varphi_\chi)=\textstyle\sum_{j\in N_0/N_1} j\beta\phi_{0,\chi}.$$

As $\phi_{0,\chi}$ is supported on $I\beta I$, if $\alpha^k j\beta\in I\beta I$ for some $k\in\mathbb{Z}$ and some $j\in N_0$, then $k=0$, by Lemma 2.5. Hence,

$$F_{0,0}(Id)=\textstyle\sum_{j\in N_0/N_1}\phi_{0,\chi}(j\beta)=\sum_{j\in N_0/N_1}1=0,$$

i.e., $F_{0,0}=0$. In all, we have verified $\psi_{n,\chi}\circ\psi_{m,\chi^s}=0$ for $n,m\geq 0$.

For $n\geq 0$ and $m=-t<0$, from Proposition 2.17 and Remark 2.18

$$\begin{aligned}\psi_{n,\chi}\circ\psi_{-t,\chi^s}&=\psi_{0,\chi}\circ(T^{\chi^s}_{-2,-1})^n\circ\psi_{-1,\chi^s}\circ(T^{\chi}_{2,3})^{t-1}\\&=\psi_{0,\chi}\circ\psi_{-1,\chi^s}\circ(T^{\chi}_{2,3})^n\circ(T^{\chi}_{2,3})^{t-1},\end{aligned}$$

hence we will be done in this case if we could show $\psi_{0,\chi}\circ\psi_{-1,\chi^s}=T^{\chi}_{2,3}$. By definition,

$$F_{0,-1}:=\psi_{0,\chi}\circ\psi_{-1,\chi^s}(\varphi_\chi)=\textstyle\sum_{j\in N_0'/N_2'} j\alpha\beta\ \phi_{0,\chi}.$$

Note that $\phi_{0,\chi}$ is supported on $I\beta I$, and if $\alpha^k j\alpha\beta \in I\beta I$ for some $k \in \mathbb{Z}$ and some $j \in N_0'$, then $k = -1$, from Lemma 2.5. Hence

$$\begin{aligned} F_{0,-1}(\alpha^{-1}) &= \sum_{j \in N_0'/N_2'} j\alpha\beta\, \phi_{0,\chi}(\alpha^{-1}) \\ &= \sum_{j=Id} \phi_{0,\chi}(\alpha^{-1} j\alpha\beta) = 1, \end{aligned}$$

where one may check $\alpha^{-1}j\alpha\beta \notin I\beta I$ for $j \in N_0' \setminus N_2'$, using trick (5).

The remaining cases could be proved in the same way and the details are omitted. □

Following [Vig04], we call $\mathscr{H}(I, \chi \oplus \chi^s)$ the second Iwahori-Hecke algebras, for a character χ of I such that $\chi \neq \chi^s$. Then, we show,

Proposition 2.22. *Let χ be a character of I such that $\chi \neq \chi^s$. Then,*

$$\mathscr{H}(I, \chi \oplus \chi^s) \cong \begin{pmatrix} \tilde{E}[X,Y]/(XY) & \tilde{E}[X] \oplus \tilde{E}[Y] \\ \tilde{E}[Y] \oplus \tilde{E}[X] & \tilde{E}[X,Y]/(XY) \end{pmatrix},$$

in which, when the isomorphism is restricted to $\mathscr{H}(I,\chi)$ (resp. $\mathscr{H}(I,\chi^s)$), it sends $T^{\chi}_{2,3}$ (resp. $T^{\chi^s}_{-2,-1}$) to X, $T^{\chi}_{-2,-1}$ (resp. $T^{\chi^s}_{2,3}$) to Y.

The algebra on the right side is denoted by $M_{X,Y}$, where the operations are those of matrices under the rule that $a_{21}a_{12} = a_{12}a_{21} = Yf_1(Y)f_2'(Y) + Xf_1'(X)f_2(X)$, if $a_{12} = (f_1'(X), f_2'(Y))$, $a_{21} = (f_1(Y), f_2(X))$.

The isomorphism depends on the order of the pair (χ, χ^s).

Proof. The second Iwahori-Hecke algebra is the space of G-homomorphisms of the direct sum of two representations of G, hence it can be written as a quasi-matrix algebra with a form in the proposition, see 2.8 of [ASS06]. Now the underlying multiplicative operations of the quasi-matrix algebra are simply translated from Proposition 2.17, Proposition 2.21, and Proposition 2.9. □

The center of $M_{X,Y}$ is $C = \{\mathrm{diag}(f,\ f),\ f \in \tilde{E}[X,Y]/(XY)\}$. Let D be the commutative sub-algebra $D = C[T_0, T_1]$ of $M_{X,Y}$, where

$$T_0 = \begin{pmatrix} 0 & (0,1) \\ (1,0) & 0 \end{pmatrix},\ T_1 = \begin{pmatrix} 0 & (1,0) \\ (0,1) & 0 \end{pmatrix}.$$

Then $M(X,Y)$ is finite over D, with two generators t_0, t_1:

$$t_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ t_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a pair $(x,y) \in \tilde{E}^2$, such that $xy = 0$, let $\chi_{(x,y)}$ be the character of C given by $\chi_{(x,y)}(\mathrm{diag}(f,f)) = f(x,y)$. We extend this to a character of D by specifying T_0 and T_1 to square roots of y and x respectively. For such an extension, say $\chi_{(x,y,\sqrt{x},\sqrt{y})}$, we form the standard module of $M_{X,Y}$: $I(x,y,\sqrt{x},\sqrt{y}) = \chi_{(x,y,\sqrt{x},\sqrt{y})} \otimes_D M_{X,Y}$.

Proposition 2.23. (1). *For $(x,y) = (0,0)$, the standard module $I(0,0)$ is the direct sum of two different characters*:

$$I(0,0) = C(1,0) \oplus C(0,1),$$

where

$$\begin{aligned} C(1,0): \ & T_i \mapsto 0, i = 0,1, \ t_0 \mapsto 1, t_1 \mapsto 0, \\ C(0,1): \ & T_i \mapsto 0, i = 0,1, \ t_0 \mapsto 0, t_1 \mapsto 1. \end{aligned}$$

(2). *For $(x,0)$, $x \neq 0$, the standard modules $I(x,0,\sqrt{x},0)$ and $I(x,0,-\sqrt{x},0)$ are simple, two dimensional and isomorphic.*

(3). *For $(0,y)$, $y \neq 0$, the standard modules $I(0,y,0,\sqrt{y})$ and $I(0,y,0,-\sqrt{y})$ are simple, two dimensional and isomorphic.*

Proof. As $M_{X,Y}$ is of rank 2 over D, the standard module $I(x,y,\sqrt{x},\sqrt{y})$ induced from a character $\chi_{(x,y,\sqrt{x},\sqrt{y})}$ of D is generated by $\chi_{(x,y,\sqrt{x},\sqrt{y})} \otimes_D t_0$ and $\chi_{(x,y,\sqrt{x},\sqrt{y})} \otimes_D t_1$. All the conclusions in the proposition can then be checked immediately by hand. □

Remark 2.24. *Any simple module of $M_{X,Y}$ with a central character has appeared in Proposition 2.23.*

The characters of $\mathscr{H}(I, \chi \oplus \chi^s)$ which correspond to $C(1,0)$ and $C(0,1)$, via the isomorphism in Proposition 2.22 are denoted respectively by

$$C_\chi(1,0) = C_{\chi^s}(0,1) \text{ and } C_\chi(0,1) = C_{\chi^s}(1,0).$$

Similarly, we denote by

$$M_\chi(x,y,\sqrt{x},\sqrt{y}) = M_{\chi^s}(y,x,\sqrt{y},\sqrt{x})$$

the simple module of $\mathscr{H}(I, \chi \oplus \chi^s)$ which corresponds to $I(x,y,\sqrt{x},\sqrt{y})$, via the isomorphism in Proposition 2.22.

2.4 The structure of $\mathscr{H}_{I_1}$

In this part, we describe the structure of $\mathscr{H}_{I_1} = \mathrm{End}_G(\mathrm{ind}_{I_1}^G 1)$, i.e., the pro-$p$ Hecke algebra of G:

The Iwahori decomposition of G leads to a double coset decomposition of G with respect to I_1:

$$G = \cup_{n \in \mathcal{N}/H_1} I_1 n I_1,$$

from which we see that the space of I_1-invariants of $\mathrm{ind}_{I_1}^G 1$ has a natural basis $\{1_{I_1 n I_1},\ n \in \mathcal{N}/H_1\}$, in which $1_{I_1 n I_1} \in \mathrm{ind}_{I_1}^G 1$ is the characteristic function of $I_1 n I_1$. Via Frobenius reciprocity, there is a basis $\{T_n,\ \ n \in \mathcal{N}/H_1\}$ of $\mathscr{H}_{I_1}$, where T_n corresponds to $1_{I_1 n I_1}$, i.e., $T_n(1_{I_1}) = 1_{I_1 n I_1}$. We would like to select some generators from these T_n.

As $\overline{H} \cong I/I_1 \cong H_0/H_1$, we will identify the characters of these groups.

For a character χ of $\overline{H}$, we define an operator e_χ in $\mathscr{H}_{I_1}$:

$$e_\chi = |\overline{H}|^{-1} \textstyle\sum_{h \in \overline{H}} \chi(h) T_h,$$

where one notes that $|\overline{H}| = -1$ in $\tilde{E}$.

Let $\varphi_\chi = e_\chi(1_{I_1})$, and as we have mentioned before, it is the function in $\mathrm{ind}_I^G \chi$, supported on I, with $\varphi_\chi(i) = \chi(i)$, for $i \in I$.

Proposition 2.25. (1). *$(e_\chi)^2 = e_\chi$; $e_\chi \cdot e_{\chi'} = 0$, if $\chi \neq \chi'$; $e_\chi(\mathrm{ind}_{I_1}^G 1) = \mathrm{ind}_I^G \chi$.*

(2). *For an element $n \in \mathcal{N} \setminus H$, we have $T_n e_\chi = e_{\chi^s} T_n$.*

(3). *If χ factors through the determinant, then*

$$T_{\beta_0}^2 e_\chi = -T_{\beta_0} e_\chi;\ \textit{otherwise},\ T_{\beta_0}^2 e_\chi = 0.$$

(4). *If $\chi = \chi^s$, then*

$$T_{\beta_1}^2 e_\chi = -T_{\beta_1} e_\chi;\ \textit{otherwise},\ T_{\beta_1}^2 e_\chi = 0.$$

Proof. (1) and (2) can be computed directly from definitions. (3) and (4) can be reduced to a special case of Theorem 4.4 in [CL76]. □

One notes that $\overline{H}$ is an abelian group of order prime to p, so the character group $\overline{H}^\wedge$ is isomorphic to $\overline{H}$. Then one can always recover each T_h from the expression of e_χ, i.e., T_h is a linear combination of all the e_χ.

Proposition 2.26. *The operators T_{β_0}, T_{β_1}, e_χ for all the characters χ of $\overline{H}$, generate $\mathscr{H}_{I_1}$ as an $\tilde{E}$-algebra.*

Proof. This comes from the following Lemma:

Lemma 2.27. (1). $\{e_\chi \cdot T_n;\ n \in \mathcal{N}/H_0,\ \chi \in \overline{H}^\wedge\}$ *is a basis of* $\mathscr{H}_{I_1}$.

(2). $T_{\alpha^n} = (T_{\beta'} \cdot T_\beta)^n$, $T_{\beta\alpha^n} = T_\beta(T_{\beta'} \cdot T_\beta)^n$, *for* $n \geq 0$.

$T_{\alpha^{-n}} = (T_\beta \cdot T_{\beta'})^n$, $T_{\beta\alpha^{-n}} = (T_{\beta'} \cdot T_\beta)^{n-1} T_{\beta'}$, *for* $n \geq 1$.

Proof. For (1), we note firstly that, $T_h T_n = T_{hn}$, $T_n T_h = T_{nh}$, for $h \in H$, $n \in \mathcal{N}$. As each T_h is an $\tilde{E}$-linear combination of e_χ, we see that the set $\{e_\chi \cdot T_n;\ n \in \mathcal{N}/H_0,\ \chi \in H^\wedge\}$ spans $\mathscr{H}_{I_1}$. To see the operators in this set are linearly independent, in view of (1) of Proposition 2.25 we are reduced to see the functions in the set $\{e_\chi \cdot T_n(1_{I_1});\ n \in \mathcal{N}/H_0\}$ are linearly independent, for a fixed character χ. This is the case, as the support of $e_\chi \cdot T_n(1_{I_1})$ is InI from the definition of T_n, and $\mathcal{N}/H_0$ is a set of representatives for the Iwahori decomposition of G, which tells us that $e_\chi \cdot T_n(1_{I_1})$ have disjoint support for $n \in \mathcal{N}/H_0$. We are done.

For (2), via the same process in Lemma 2.2, one can check similar induction relations hold in this case. Then the result comes. □

Let h_0 be the matrix $\mathrm{diag}(-t^{-1}, 1, t)$, and T_{h_0} has the inverse $T_{h_0^{-1}}$ in $\mathscr{H}_{I_1}$. Then $\beta_0 = h_0\beta$, $\beta_1 = h_0\beta'$. Hence $T_{\beta_0} = T_{h_0}T_\beta$, $T_{\beta_1} = T_{h_0}T_{\beta'}$. In view of the above Lemma, one see the Proposition is true. □

Remark 2.28. *Let* π *be a smooth representation of* G. *Let* v *be a non-zero element in* π^{I_1}. *We see the right action of* T_{h_0} *on* v *is*

$$v|T_{h_0} = \sum_{j \in I_1/I_1 \cap h_0^{-1} I_1 h_0} j h_0^{-1} \cdot v = h_0^{-1} v.$$

Proposition 2.29. *There is an isomorphism of* $\tilde{E}$*-algebras*:

$$\mathscr{H}_{I_1} \cong \oplus_{\chi = \chi^s} \mathscr{H}(I, \chi) \oplus_{\chi \neq \chi^s} \mathscr{H}(I, \chi \oplus \chi^s),$$

Proof. As $\mathrm{ind}_{I_1}^I 1 = \oplus_\chi \chi$, we see that $\mathrm{ind}_{I_1}^G 1 = \oplus_\chi \mathrm{ind}_I^G \chi$, where χ goes through the characters of the group $\overline{H} \cong I/I_1$. For two such characters χ and χ', from the Frobenius reciprocity and the decomposition of restriction of induced representation to subgroup, we can conclude that $\mathrm{Hom}_G(\mathrm{ind}_I^G \chi, \mathrm{ind}_I^G \chi') \neq 0$ if and only if $\chi' = \chi$ or $\chi' = \chi^s$. Then the Proposition follows. □

Theorem 2.30. *The simple modules listed in Proposition 2.14, Proposition 2.16, and Proposition 2.23, give all that of* $\mathscr{H}_{I_1}$.

2.5 Non-supersingular modules of $\mathscr{H}_{I_1}$

Given a character ε of the standard Borel subgroup B of G and $\varepsilon_0 = \chi_1 \otimes \chi_2$ be the restriction of ε to H_0. Let $\langle g_1,\ g_2\rangle_{\tilde{E}}$ be the basis of $(\mathrm{ind}_B^G\,\varepsilon)^{I_1}$, in which g_1 and g_2 are respectively supported on BI_1 and $B\beta I_1$, satisfying that $g_1(Id) = g_2(\beta) = 1$. This space admits a natural right action of $\mathscr{H}_{I_1}$.

Proposition 2.31. *The right action of $\mathscr{H}_{I_1}$ on $(\mathrm{ind}_B^G\,\varepsilon)^{I_1}$ is as follows:*

$(g_1,\ g_2)T_\beta = (g_1,\ g_2)\begin{pmatrix} 0 & 0 \\ 1 & a_{22}\end{pmatrix}$, *where $a_{22} = -\chi_1'(-1)$ or 0, depending on whether ε_0 factors through the determinant or not;*

$(g_1,\ g_2)T_{\beta'} = (g_1,\ g_2)\begin{pmatrix} a_{11} & \varepsilon(\alpha) \\ 0 & 0\end{pmatrix}$, *where $a_{11} = -\chi_1'(-1)$ or 0, depending on whether $\varepsilon_0 = \varepsilon_0^s$ or not.*

$(g_1,\ g_2)e_\chi = (g_1,\ g_2)\begin{pmatrix} a_{11}(\chi) & 0 \\ 0 & a_{22}(\chi)\end{pmatrix}$, *where $a_{11}(\chi) = 1$ if $\chi = \varepsilon_0$; otherwise $a_{11}(\chi) = 0$. $a_{22}(\chi) = 1$ if $\chi = \varepsilon_0^s$; otherwise $a_{22}(\chi) = 0$.*

Proof. The action of e_χ on f_i is

$$g_i|e_\chi = |\overline{H}|^{-1}\textstyle\sum_{h\in\overline{H}}\ \chi(h)h^{-1}g_i,$$

and the result follows directly from valuating these functions at Id and β.

From Proposition 6 in [BL94], we have

$$g_i|T_\beta = \sum_{j\in I_1/I_1\cap\beta I_1\beta} j\beta\cdot g_i = \textstyle\sum_{j\in N_0/N_1}\ j\beta\cdot g_i,$$
$$g_i|T_{\beta'} = \sum_{j\in I_1/I_1\cap\beta' I_1\beta'} j\beta'\cdot g_i = \textstyle\sum_{j\in N_1'/N_2'}\ j\beta'\cdot g_i.$$

Case 1, $\varepsilon_0 = \varepsilon_0^s$,

Certainly $\sum_{j\in N_0/N_1} j\beta\cdot g_1(Id) = 0$, as $g_1(j\beta) = 0$ for all the $j \in N_0/N_1$. For $\sum_{j\in N_0/N_1} j\beta\cdot g_1(\beta)$, one see the first term for $j \in N_1$ is equal to 1. To see the remaining terms are all zero, for a $j = n(x, y) \in N_0\backslash N_1$, i.e., $y \in U_E$, we have from Lemma 1.2 that:

$$\beta n(x,y)\beta = n(\bar{y}^{-1}x, y^{-1})\mathrm{diag}(\bar{y}^{-1}, -y^{-1}\bar{y},\ y)\beta i, \text{ for some } i \in N_0,$$

and this gives the result. Hence, $g_1|T_\beta = g_2$.

$\sum_{j\in N_0/N_1} j\beta\cdot g_2(Id)$ is also zero, as every term is 1 and $\#N_0/N_1 = q^3$. The term for $j \in N_1$ in $\sum_{j\in N_0/N_1} j\beta\cdot g_2(\beta)$ is zero. From the above identity, we get

$$\sum_{j\in(N_0\setminus N_1)/N_1} j\beta\cdot g_2(\beta) = \sum_{n(x,y)\in(N_0\setminus N_1)/N_1} \varepsilon_0(\mathrm{diag}(\bar{y}^{-1}, -y^{-1}\bar{y},\, y))$$
$$= \sum_{n(x,y)\in(N_0\setminus N_1)/N_1} \chi_1(\bar{y}^{-1})\chi_2(-y^{-1}\bar{y}),$$

where we write $\varepsilon_0 = \chi_1\otimes\chi_2$, and χ_1, χ_2 are respectively characters of $k_E^\times$ and k_E^1. This is the sum we have dealt with in the argument of Proposition 2.6. By Remark 1.4, the condition $\varepsilon_0 = \varepsilon_0^s$ implies there is a character χ_1' of k_E^1, such that $\chi_1(x) = \chi_1'(x\bar{x}^{-1})$ for $x\in k_E^\times$. Then, if ε_0 factors through the determinant,

$$\textstyle\sum_{j\in(N_0\setminus N_1)/N_1} j\beta\cdot g_2(\beta) = -\chi_1'(-1);$$

otherwise,

$$\textstyle\sum_{j\in(N_0\setminus N_1)/N_1} j\beta\cdot g_2(\beta) = 0,$$

In summary, $g_2|T_\beta = 0$ or $-\chi_1'(-1)g_2$. We have shown the first half in **Case 1**.

The term for $j\in N_2'$ in $\sum_{j\in N_1'/N_2'} j\beta'\cdot g_2(Id)$ is $\varepsilon(\alpha)$, the other terms are all zero (see the identity for such a $j\beta'$ below). On the other hand, $\sum_{j\in N_1'/N_2'} j\beta'\cdot g_2(\beta) = 0$ ($\beta j\beta'\in B$ for $j\in N_1'/N_2'$). Hence $g_2|T_{\beta'} = \varepsilon(\alpha)g_1$

Every term in the sum $\sum_{j\in N_1'/N_2'} j\beta'\cdot g_1(\beta)$ is $\varepsilon(\alpha^{-1})$, and the sum is zero as $\#N_1'/N_2' = q$. The term for $j\in N_2'$ in the sum $\sum_{j\in N_1'/N_2'} j\beta'\cdot g_1(Id)$ is zero. For a $n'(0,\varpi_E y)\in N_1'\setminus N_2'$, we have

$$j\beta' = n_1\mathrm{diag}(\bar{y}^{-1}, -y^{-1}\bar{y},\, y)n_2, \text{ for some } n_1\in N_{-1},\ n_2\in N_1',$$

therefore,

$$\textstyle\sum_{j\in N_1'\setminus N_2'} j\beta'\cdot g_1(Id) = \sum\limits_{y\in L_1^*} \chi_1(\bar{y}^{-1}),$$

which is equal to $-\chi_1'(-1)$ as we know. We have shown $g_1|T_{\beta'} = -\chi_1'(-1)g_1$.

Case 2, $\varepsilon_0\neq\varepsilon_0^s$,

in view of **Case 1**, we are left to show the following two sums both vanish:

$$\sum_{j\in N_0/N_1} j\beta\cdot g_2(\beta) = \sum_{n(x,y)\in N_0\setminus N_1} \chi_1(\bar{y}^{-1})\chi_2(-y^{-1}\bar{y}),$$
$$\sum_{j\in N_1'/N_2'} j\beta'\cdot g_1(Id) = \sum_{y\in L_1^*} \chi_1(\bar{y}^{-1}).$$

For the second sum, in the notation of 1.1, it's equal to

$$\sum_{l=0}^{q-2} \chi_1(t)^{2l+1},$$

where $t = [u]^{\frac{q+1}{2}}$ and u is a generator of $k_E^\times$, defined in section 1.2. However the condition $\varepsilon_0 \neq \varepsilon_0^s$ is equivalent to $\chi_1(t)^2 \neq 1$. Then the above sum is zero, as $t^{2(q-1)} = 1$.

For the first sum, it can be decomposed into two parts: the first part for which x vanishes is just the second sum. The remaining part of the first sum is then reduced to

$$\sum_{\mathrm{tr}(y)\neq 0} \chi_1(\bar{y}^{-1})\chi_2(-y^{-1}\bar{y}) = \sum_{\mathrm{tr}(y)\neq 0} \chi_1(y)\chi_2(-y^{-1}\bar{y}).$$

We note that $\mathrm{tr}(y) \neq 0$ means $-y^{-1}\bar{y} \neq 1$. For a $x = u^{(q-1)l} \in k_E^1, \neq 1$, for some l, $1 \leq l \leq q$, the solutions y of the equation $-y^{-1}\bar{y} = x$ (over the finite field k_E) are $u^{l+\frac{q+1}{2}} \cdot u^{(q+1)m}$, $m = 0, \cdots, q-2$. We can then rewrite the above sum as:

$$\sum_{l=1}^{q} \chi_2(u^{(q-1)l}) \sum_{m=0}^{q-2} \chi_1(u^{l+\frac{q+1}{2}+(q+1)m})$$

which is just

$$\chi_1(t) \sum_{l=1}^{q} \chi_2(u^{(q-1)l}) \cdot \chi_1(u^l) \sum_{m=0}^{q-2} \chi_1(u^{(q+1)m}).$$

The condition $\chi_1(u^{q+1}) \neq 1$ tells the inner sum of the above is zero. We are done. □

Corollary 2.32. *The modules $(\mathrm{ind}_B^G\, \varepsilon)^{I_1}$ of $\mathscr{H}_{I_1}$ in Proposition 2.31 are reducible, if and only if $\varepsilon(\alpha) = 1$ and ε_0 factors through the determinant, i.e., ε factors through the determinant.*

Proof. This can be verified directly, in light of the above Proposition. □

2.6 Supersingular characters

In light of the results above, we would like to select those simple modules of $\mathscr{H}_{I_1}$ which does not appear in the I_1-invariants of principal series.

Definition 2.33. *Let χ be a character of I, satisfying $\chi = \chi^s$. Let $\langle v \rangle_{\bar{E}}$ be a one-dimensional vector space, on which we define a right $\mathscr{H}_{I_1}$-module:*

(1) *Suppose χ factors through the determinant,*

$$C_1: \quad v \cdot e_\chi = v,\ v \cdot T_{\beta_0} = 0,\ v \cdot T_{\beta_1} = -v;$$
$$C_2: \quad v \cdot e_\chi = v,\ v \cdot T_{\beta_0} = -v,\ v \cdot T_{\beta_1} = 0.$$

(2) *Suppose χ does not factor through the determinant,*

$$C_3: \quad v \cdot e_\chi = v,\ v \cdot T_{\beta_0} = 0,\ v \cdot T_{\beta_1} = 0;$$
$$C_4: \quad v \cdot e_\chi = v,\ v \cdot T_{\beta_0} = 0,\ v \cdot T_{\beta_1} = -v.$$

Definition 2.34. *Let χ be a character of I, satisfying $\chi \neq \chi^s$. Let $\langle v \rangle_{\tilde{E}}$ be a one-dimensional vector space, on which we define a right $\mathscr{H}_{I_1}$-module*:

$$\begin{aligned} C_5: &\quad v \cdot e_\chi = v,\ v \cdot e_{\chi^s} = 0,\ v \cdot T_{\beta_0} = 0,\ v \cdot T_{\beta_1} = 0; \\ C_6: &\quad v \cdot e_\chi = 0,\ v \cdot e_{\chi^s} = v,\ v \cdot T_{\beta_0} = 0,\ v \cdot T_{\beta_1} = 0. \end{aligned}$$

One calls the characters defined in Definition 2.33, 2.34 *supersingular*, as they are the simple modules of $\mathscr{H}_{I_1}$ which are *not* from the I_1-invariants of principal series of G, according to Proposition 2.31 and Theorem 2.30.

3 The compactly induced representation $\mathrm{ind}_{K_0}^G \sigma$

The content of this chapter is to understand the compactly induced representation $\mathrm{ind}_{K_0}^G \sigma$ for any irreducible smooth representation σ of K_0, for which we have mainly arrived at the following:

We first describe the structure of the spherical Hecke algebra $\mathscr{H}(K_0, \sigma)$, which is the content of section 3.1. The main result is Corollary 3.4.

We next move to describe the action of $\mathscr{H}_{I_1}$ on the I_1-invariants of $\mathrm{ind}_{K_0}^G \sigma$, which is the main focus of 3.2. Especially, we prove a codimension result, Proposition 3.11, which is an analogue of a result of Barhel–Livné on $GL_2(F)$ and crucial to many later arguments of this thesis.

Another major part of this chapter is an explicit formula for the Hecke operator T, Lemma 3.12 and Proposition 3.13, which takes space in 3.3. As an application, we show $\mathrm{ind}_{K_0}^G \sigma$ is free over $\mathscr{H}(K_0, \sigma)$ when σ is a character $\chi \cdot \det$ of K_0, or a twist of St by a character: $\chi \cdot \det \otimes St$, where St is the inflation of the Steinberg representation of $\overline{G}(k_F)$. We indeed conjecture this holds for any irreducible smooth representations σ of K_0 and prove it under a natural assumption. However, based on results we have proved in section 3.1, it is immediate to show that a compromise result holds (Proposition 3.18): $\mathrm{ind}_{K_0}^G \sigma$ is faithfully flat over $\mathscr{H}(K_0, \sigma)$.

In 3.5, we describe the G-Hom space from $\mathrm{ind}_{K_0}^G \sigma$ to a principal series, and show that it is at most one-dimensional[5]. Then we determine the eigenvalue of the natural action of $\mathscr{H}(K_0, \sigma)$ on that space.

In the remaining part of this chapter, we first provide more information in 3.6 for the Bruhat–Tits tree $\triangle$ of G, which is used essentially in Chapter 4.

We end this chapter in 3.7 by some computations on the behaviour of the I_1-invariants of $\mathrm{ind}_{K_0}^G \sigma$ under the action of the operator T.

[5]This has been proved firstly in [Abd11].

3.1 $\mathscr{H}(K_0,\sigma)$ as a polynomial ring in one variable

Let (σ, W) be an irreducible smooth representations of K_0, $\mathrm{ind}_{K_0}^G\sigma$ be the compactly induced smooth representation, i.e., the representation of G with underlying space $S(G,\sigma)$

$$S(G,\sigma) = \{f : G \to W : f(kg) = \sigma(k)\cdot f(g), \text{ for any } k \in K_0 \text{ and } g \in G, \text{ locally constant with compact support}\}$$

and G acting by right translation.

Denote by $\mathscr{H}(K_0,\sigma)$ the endomorphism algebra $\mathrm{End}_G(\mathrm{ind}_{K_0}^G\sigma)$.

From Proposition 5 in [BL94], one has $\mathscr{H}(K_0,\sigma) \cong \mathscr{H}_{K_0}(\sigma)$, where $\mathscr{H}_{K_0}(\sigma)$ is the algebra defined as:

$$\mathscr{H}_{K_0}(\sigma) = \{f : G \to \mathrm{End}(W) : f(kgk') = \sigma(k)f(g)\sigma(k'), \text{ for any } k,k' \in K_0 \text{ and } g \in G, \text{locally constant with compact support}\}$$

where the multiplication is defined by convolution: for $h_1, h_2 \in \mathscr{H}_{K_0}(\sigma)$

$$h_1 * h_2(x) = \textstyle\sum_{g\in G/K_0} h_1(g)h_2(g^{-1}x), \text{ for } x \in G.$$

As K_0^1 acts trivially on the representation σ and $\overline{G}(k_F) \cong K_0/K_0^1$, we identify σ with the inflation of an irreducible representation $\overline{G}(k_F)$. As usual, denote by σ^{N_0} the subspace of N_0-invariant of σ, and by σ_{N_0} the quotient of σ by the subspace $\sigma(N_0)$ generated by the set $\{u\cdot v - v : u \in N_0, v \in \sigma\}$. We use similar notations for $\sigma^{N_0'}$ and $\sigma_{N_0'}$.

Lemma 3.1. *σ^{N_0} and $\sigma_{N_0'}$ are both one-dimensional. Furthermore, the image of σ^{N_0} is non-trivial in $\sigma_{N_0'}$, via the natural composition $j_\sigma^*\colon \sigma^{N_0} \hookrightarrow \sigma \to \sigma_{N_0'}$. Furthermore, we have*

$$\sigma = \sigma^{N_0} \oplus \sigma(N_0').$$

Proof. See [CE04], Theorem 6.12. □

Let j_σ be the linear map in $\mathrm{Hom}(\sigma_{I_1'}, \sigma^{I_1})$, which is the inverse of j_σ^* described in Lemma 3.1. Especially, we see $j_\sigma(\bar{v}) = v$ for $v \in \sigma^{I_1}$ and it vanishes on $\sigma(I_1')$.

One notes that there is a unique constant $\lambda_{\beta,\sigma} \in \tilde{E}$, independent of the choice of non-zero $v \in \sigma^{I_1}$, such that

$$\sigma(\beta)\cdot v - \lambda_{\beta,\sigma} v \in \sigma(I_1').$$

Remark 3.2. *It is known from a recent preprint ([HV12], Proposition 3.17) of Henniart and Vignéras that $\lambda_{\beta,\sigma}$ is non-zero if and only if σ is one-dimensional. In fact, it is directly to verify $\sigma = St$, see Proposition 4.14.*

We record the value of $\lambda_{\beta,\sigma}$ as follows

$$\lambda_{\beta,\sigma} = \begin{cases} \eta(-1), & \text{if } \sigma = \eta \circ \det, \\ 0, & \text{otherwise.} \end{cases}$$

Recall the double coset decomposition of G with respect to K_0, say $G = \cup_{n\geq 0} K_0\alpha^n K_0$.

Let φ be a function in $\mathscr{H}_{K_0}(\sigma)$, supported on the double coset $K_0\alpha^n K_0$. Then, for any $k_1,\ k_2 \in K_0$, satisfying $k_1\alpha^n = \alpha^n k_2$, we are given $\sigma(k_1)\varphi(\alpha^n) = \varphi(\alpha^n)\sigma(k_2)$. When $n = 0$, $\varphi(Id)$ commutes with all $\sigma(k)$. As σ is irreducible, we must have that $\varphi(Id)$ is a scalar.

For $n > 0$, let $k_1 = n'(\varpi_E^n x, \varpi_E^{2n} y), x, y \in \mathfrak{o}_E$, then $\sigma(k_1) = 1$, as $k_1 \in K_0^1$. And now $k_2 = \alpha^{-n}k_1\alpha^n = n'(x,y)$. Hence, $\varphi(\alpha^n) = \varphi(\alpha^n)\cdot\sigma(n'(x,y))$. We see $\varphi(\alpha^n)$ factorizes through $\sigma_{I_1'}$. Similarly, for $k_1 = n(x,y), x, y \in \mathfrak{o}_E$, we get $\sigma(n(x,y))\varphi(\alpha^n) = \varphi(\alpha^n)$, which is to say that $\mathrm{Im}(\varphi(\alpha^n)) \in \sigma^{I_1}$. In other words, $\varphi(\alpha^n)$ should only differ from j_σ a scalar. Then we are led to:

For $n \geq 0$, let φ_n be the function in $\mathscr{H}_{K_0}(\sigma)$, supported on $K_0\alpha^n K_0$, determined by its value on α^n: $\varphi_0(Id) = Id_W,\ \varphi_n(\alpha^n) = j_\sigma,\ n > 0$.

Proposition 3.3. *$\{\varphi_n\}_{n\geq 0}$ consists of a basis of $\mathscr{H}_{K_0}(\sigma)$, and they satisfy the following convolution relations: for $n \geq 1,\ l \geq 0$,*

$$\varphi_1 * \varphi_n(\alpha^l) = \begin{cases} 0, & l \neq n,\ n+1; \\ c\cdot j_\sigma, & l = n; \\ j_\sigma, & l = n+1, \end{cases} \tag{7}$$

where c is some constant in $\tilde{E}$, dependent on σ (the exact value of c is given in the final part of the proof).

Proof. By definition, for $n \geq 1,\ l \geq 0$,

$$\varphi_1 * \varphi_n(\alpha^l) = \textstyle\sum_{g\in G/K_0} \varphi_1(g)\varphi_n(g^{-1}\alpha^l).$$

As the support of φ_1 is $K_0\alpha K_0 = \cup_{k\in K_0/K_0\cap\alpha K_0\alpha^{-1}} k\alpha K_0$, the sum becomes

$$\begin{aligned} &\textstyle\sum_{k\in K_0/K_0\cap\alpha K_0\alpha^{-1}} \varphi_1(k\alpha)\varphi_n(\alpha^{-1}k^{-1}\alpha^l) \\ &= \textstyle\sum_{k_1\in K_0/I} \sum_{k_2\in N_1'/N_2'} \varphi_1(k_1k_2\alpha)\varphi_n(\alpha^{-1}k_2^{-1}k_1^{-1}\alpha^l). \end{aligned}$$

For further calculation, we split the above sum into two parts, say,

$$\textstyle\sum_1 = \displaystyle\sum_{k_2\in N_1'/N_2'} \varphi_1(\beta k_2\alpha)\varphi_n(\alpha^{-1}k_2^{-1}\beta\alpha^l)$$

and

$$\textstyle\sum_2 = \sum_{k_1\in N_0'/N_1'} \sum_{k_2\in N_1'/N_2'} \varphi_1(k_1k_2\alpha)\varphi_n(\alpha^{-1}k_2^{-1}k_1^{-1}\alpha^l).$$

Then we claim $\sum_1$ is always 0. We go into more detail in the following: $\sum_1$ could be simplified as

$$\sum_1 = \sum_{k_2\in N_1'/N_2'} \sigma(\beta) j_\sigma \varphi_n(\alpha^{-1}k_2^{-1}\beta\alpha^l).$$

We note at first that $\alpha^{-1}k_2^{-1}\beta\alpha^l \in K_0\alpha^{-(l+1)}K_0$, hence we only need to consider the case when $l+1=n$. In this case from the definition of φ_n, the sum $\sum_1$ is reduced to

$$\sum_1 = \sum_{k_2} \sigma(\beta) j_\sigma \sigma(\beta) j_\sigma,$$

which is clearly zero, as it is counted q times.

For the remaining $\sum_2$, we note the part $\sum_1'$ for which $k_1 \in N_0' \setminus N_1'$ is equal to 0. A simple calculation using (3) gives

$$\alpha^{-1}k_2^{-1}k_1^{-1}\alpha^l = k'\alpha^{-(l+1)}k'',$$

where $k' \in N_1$ and $k'' \in K_0$. As a result, when $l \neq n-1$, $\sum_1' = 0$. When $l = n-1$, one can re-write $\sum_1'$ as

$$\sum_1' = \sum_{k_2} (\sum_{k_1} f'),$$

where f' is now a function only related to k_1. As the inner sum of $\sum_1'$ is counted q times, it is zero. For the concrete form of f', one needs to distinguish $l=0$ or not, and we don't record it here as it is not necessary.

The other part depends:

$$\sum_{k_2\in N_1'/N_2'} \varphi_1(k_2\alpha)\varphi_n(\alpha^{-1}k_2^{-1}\alpha^l) = \begin{cases} j_\sigma, & l = n+1, \\ c\cdot j_\sigma, & l = n, \\ 0, & \text{otherwise}, \end{cases}$$

where c is the constant in $\tilde{E}$ determined from the following:

$$\sum_{y\in L_1^*} j_\sigma \cdot \sigma(\beta)\cdot j_\sigma \cdot \sigma(\mathrm{diag}(y,1,-y^{-1})) = c\cdot j_\sigma.$$

From the definition of φ_1, the above sum, denoted by $\sum_2''$, is reduced to

$$\sum_2'' = \sum_{k_2} j_\sigma \varphi_n(\alpha^{-1}k_2^{-1}\alpha^l).$$

We treat an exceptional case first: when $l=0$, each term in $\sum_2''$ is non-zero only if $n=1$; but in this case, the sum itself is clearly zero. We assume $l \geq 1$. For the term $k_2 \in N_2'$, we get $j_\sigma\varphi_n(\alpha^{l-1})$, where we note that j_σ is trivial on $\sigma(I_1')$. Hence, it is non-zero if and only if $l = n+1$ and in this case it is equal to $j_\sigma^2 = j_\sigma$. For the remaining terms in $\sum_2''$, we use (3) again

$$\alpha^{-1}n'(0,\varpi_E\bar{y})\alpha^l = n(0,\varpi_E\bar{y}^{-1})\alpha^{-l}\mathrm{diag}(y^{-1},1,\bar{y})u'\beta,$$

for some $u' \in N_1'$. Therefore, each term is non-zero if and only if $l = n$, and in that case, we get

$$\textstyle\sum_{y\in L_1^*} j_\sigma \cdot \sigma(\beta) \cdot j_\sigma \cdot \sigma(\mathrm{diag}(\bar{y}, 1, y^{-1})),$$

which should be nothing but a copy of j_σ, say $c \cdot j_\sigma$.

Let $\chi = \chi_\sigma$ be the character of I, which is determined by σ, from its action on σ^{I_1}. Then, we can make more clear about c :

$$c = \begin{cases} -\lambda_{\beta,\sigma} \cdot \chi_1'(-1), & \text{if } \chi = \chi^s, \\ 0, & \text{otherwise.} \end{cases}$$

We are done. □

From the above Proposition, we see immediately that $\varphi_1 * \varphi_n = c\cdot\varphi_n + \varphi_{n+1}$. Let T_n be the operator in $\mathscr{H}(K_0,\sigma)$ which corresponds to φ_n, via $\mathcal{L}$, and put $T = T_1$. Then we have,

Corollary 3.4. *$\mathscr{H}(K_0,\sigma)$ is isomorphic to $\tilde{E}[T]$.*

We will use the following variant of last Proposition in some later argument.

Corollary 3.5. *Let σ and σ' be two irreducible smooth representations of K_0. Denote by T and T' respectively the Hecke operators in $\mathscr{H}(K_0,\sigma)$ and $\mathscr{H}(K_0,\sigma')$, defined from Corollary 3.4. Then we have*

(1). The space $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_{K_0}^G\sigma') \neq 0$ iff $\chi_\sigma = \chi_{\sigma'}$.

(2). Assume the condition in (1) is satisfied. Then, the natural $\mathscr{H}(K_0,\sigma')-\mathscr{H}(K_0,\sigma)$-bi-module structures coincide in the following sense. More precisely, there exists a unique constant $c_{\sigma,\sigma'}$, and for any $L \in \mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_{K_0}^G\sigma')$ and all polynomial f

$$L \circ f(T) = f(T' + c_{\sigma,\sigma'}) \circ L$$

Proof. As in the case $\sigma = \sigma'$, one can identify $\mathscr{H}(K_0,\sigma')$-$\mathscr{H}(K_0,\sigma)$-bi-module $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_{K_0}^G\sigma')$ with the $\mathscr{H}_{K_0}(\sigma')$-$\mathscr{H}_{K_0}(\sigma)$-bi-module $\mathscr{H}_{K_0}(\sigma,\sigma')$. Then (1) follows from a variant of remarks before Proposition 3.3.

From the argument of (1), we get a $\tilde{E}$-basis $\{\varphi_n\}_{n\geq 0}$ for $\mathscr{H}_{K_0}(\sigma,\sigma')$, on which the left action of $\mathscr{H}_{K_0}(\sigma')$ is described by

$$\phi_{\sigma',1} * \varphi_n = c_{\sigma'} \cdot \varphi_n + \varphi_{n+1}$$

where $\phi_{\sigma',1} \in \mathscr{H}_{K_0}(\sigma')$, i.e., the φ_1 in last Proposition, whose calculations could be tracked by the argument of Proposition 3.3. The $*$ is the natural convolution defined in the same way as usual, see [BL94] for more details. Similarly, we have

$$\varphi_n * \phi_{\sigma,1} = c_\sigma \cdot \varphi_n + \varphi_{n+1}$$

Now we put $c_{\sigma,\sigma'} = c_\sigma - c_{\sigma'}$, and then (2) follows. □

Remark 3.6. *The idea to consider the bi-module structures above is due to Florian Herzig, who has proved such result in a very general situation, as a variant of his Satake embedding.*

3.2 $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$ as an $\mathscr{H}_{I_1}$-module

The group I acts on σ^{I_1} as a character, and from now on we denote it by χ_σ.

Recall we have a double coset decomposition $G = \cup_{n\in\mathbb{Z}} K_0\alpha^n I_1$. From [BL94], we see $(\mathrm{ind}_{K_0}^G\sigma)^{I_1} = \{f \in S(G,\sigma);\ f(kgi) = \sigma(k)f(g),\ \text{for } k \in K_0,\ g \in G,\ i \in I_1\}$. Let f be a function in $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$, supported in $K_0\alpha^n I_1$. For $k \in K_0,\ i \in I_1$ such that $k\alpha^n = \alpha^n i$, $f(\alpha^n)$ should satisfy $\sigma(k)f(\alpha^n) = f(\alpha^n)$.

For $n \geq 0$, and $u = n(x,y), x,y \in \mathfrak{o}_E$, we get $\sigma(u)f(\alpha^n) = f(\alpha^n)$, which means $f(\alpha^n)$ is fixed by N_0. Similarly, for negative n, and $u' = n'(x,y), x,y \in \mathfrak{o}_E$, we find $\sigma(u')f(\alpha^n) = f(\alpha^n)$, i.e., $f(\alpha^n)$ is fixed by N_0'. We note that $\sigma^{I_1} = \sigma^{N_0}$ and $\sigma^{I_1'} = \sigma^{N_0'}$, as $I_1 = N_0\cdot(I_1\cap B')$ and $I_1' = N_0'\cdot(I_1'\cap B)$.

Choose and fix a non-zero $v_0 \in \sigma^{I_1}$. Put $v_0' = \beta v_0$. Let f_n be the function in $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$, supported on $K_0\alpha^{-n}I_1$, such that

$$f_n(\alpha^{-n}) = \begin{cases} v_0', & n > 0, \\ v_0, & n \leq 0. \end{cases} \tag{8}$$

Proposition 3.7. (1). *$\{f_n\}$ consists of a basis of the space $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$;*

(2). *The action of I on $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$ is as follows: For $i \in I$, let h be the element in H_0 such that $iI_1 = hI_1$. Then,*

$$i \cdot f_n = \begin{cases} \chi_\sigma(h)\cdot f_n, & n \leq 0, \\ \chi_\sigma^s(h)\cdot f_n, & n > 0. \end{cases} \tag{9}$$

Proof. The first part comes from the description before the Proposition, and the second part can be checked easily. □

Corollary 3.8. *Let χ_σ be the character of I on σ^{I_1}, then we have*

$$(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi_\sigma} = \begin{cases} \langle f_n, n\in\mathbb{Z}\rangle, & \textit{if } \chi_\sigma = \chi_\sigma^s; \\ \langle f_{-n}, n \geq 0\rangle, & \textit{if } \chi_\sigma \neq \chi_\sigma^s. \end{cases} \tag{10}$$

$$(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi_\sigma^s} = \langle f_n, n > 0\rangle, \quad \textit{if } \chi_\sigma \neq \chi_\sigma^s. \tag{11}$$

We turn to describe the right actions of the Iwahori-Hecke algebras on $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$. Recall we have determined the structure of $\mathscr{H}(I,\chi)$, for any character χ of I, say Proposition 2.6 and Proposition 2.9.

Proposition 3.9. *Suppose $\chi=\chi^s$. Then,*

(1)[6]. $f_0|T_{0,-1}=c'\cdot f_0,\ f_0|T_{2,1}=f_1$.

(2). *For $n>0$,*

$$f_n|T_{0,-1}=f_{-n},\ f_{-n}|T_{0,-1}=-f_{-n};$$
$$f_n|T_{2,1}=-f_n,\ f_{-n}|T_{2,1}=f_{n+1}.$$

Proof. (1). By definition, we have

$$f_0|T_{0,-1}=\sum_{i\in N_0/N_1}i\beta f_0,$$

which is a function supported in K_0 and hence equals $c'f_0$ for some c'; in other words $\sum_{i\in N_0/N_1}i\beta v_0=c'v_0$. Taking the quadratic relation of $T_{0,-1}$ into account, we see immediately that $c'=0$ when χ does not factor through the determinant. In the case that χ factors through the determinant, we see $c'=0$ or $-\chi_1'(-1)$. A little explicit calculation shows that $c'=0$ if σ is a character (this is indeed clear), otherwise $c'=-\chi_1'(-1)$. In summary, we have

$$c'=\begin{cases}-\eta(-1), & \text{if } \sigma=\eta\circ\det\otimes St,\\ 0, & \text{otherwise.}\end{cases}$$

For the second relation in (1), we also have by definition that

$$f_0|T_{2,1}=\sum_{i\in N_1'/N_2'}i\alpha\beta\cdot f_0,$$

which is supported in $K_0\alpha^{-1}I_1$. Its value at α^{-1} is just v_0', as one can check as follows:

$$\sum_{i\in N_1'/N_2'}f_0(\alpha^{-1}i\alpha\beta)=\sum_{i=Id}\beta f_0(Id)=v_0',$$

where we note that $\alpha^{-1}i\alpha\beta\in I\alpha^{-1}I$ for $i\in N_1'\setminus N_2'$. The result follows.

When $n>0$, for the formulas in (2), we will check $f_n\mid T_{0,-1}=f_{-n}$ and $f_{-n}\mid T_{2,1}=f_{n+1}$ in detail. Note that $f_n\mid T_{0,-1}$ and $f_{-n}\mid T_{2,1}$ are both I_1-invariant and supported on a union of double cosets of the form $K_0\alpha^kI_1$.

By definition,

$$f_n\mid T_{0,-1}=\sum_{i\in N_0/N_1}i\beta f_n.$$

[6] the exact value of c' is given in the proof.

As f_n is supported on $K_0\alpha^{-n}I = I\alpha^{-n}I \cup I\beta\alpha^{-n}I$, $\alpha^k i\beta \in I\beta\alpha^{-n}I$ for some $k \in \mathbb{Z}$ and $i \in N_0$ forces $k = n$, using Lemma 2.5; also, $\alpha^{k'} i\beta \in I\alpha^{-n}I$ for some $k' \in \mathbb{Z}$ and $i \in N_0$ implies immediately that $i \in N_0 \setminus N_1$ and $k' > 0$, where then a contradiction arises after applying Lemma 1.2 to i. Hence,

$$\textstyle\sum_{i\in N_0/N_1} i\beta f_n(\alpha^n) = \sum_{i=Id} f_n(\alpha^n\beta) = v_0',$$

where we check that $\alpha^n i\beta \notin I\alpha^{-n}I$ for $i \in N_0 \setminus N_1$: applying Lemma 1.2 to $\alpha i\alpha^{-1}$, the situation is reduced to a contradiction from Lemma 2.5.

Also from definition,

$$f_{-n} \mid T_{2,1} = \textstyle\sum_{i\in N_1'/N_2'} i\alpha\beta f_{-n}.$$

Recall f_{-n} is supported on $K_0\alpha^n I$, and $K_0\alpha^n I = I\alpha^n I \cup I\beta\alpha^n I$. It is clear that $\alpha^k i\alpha\beta \in I\alpha^n I$, for some $i \in N_1'$ and $k \in \mathbb{Z}$, implies $i \in N_1' \setminus N_2'$ and $k < 0$, where a contradiction then arises after applying Lemma 1.2 to $\alpha^{-1}i\alpha$. Now, $\alpha^k i\alpha\beta \in I\beta\alpha^n I$ for some $k \in \mathbb{Z}$ and some $i \in N_1'$ forces $k = -(n+1)$, using Lemma 2.5. Now

$$\textstyle\sum_{i\in N_1'/N_2'} i\alpha\beta f_{-n}(\alpha^{-(n+1)}) = \sum_{i=Id} f_{-n}(\alpha^{-(n+1)}\alpha\beta) = v_0',$$

where we check that $\alpha^{-(n+1)} i\alpha\beta \notin I\beta\alpha^n I$ for $i \in N_1' \setminus N_2'$, using the trick (5) in the argument of Proposition 2.17. We are done. ☐

Proposition 3.10. *Suppose $\chi_\sigma \neq \chi_\sigma^s$. Write χ for χ_σ for short. Then,*

(1) *for $n \geq 0$,*

$$f_{-n}|T_{2,3}^{\chi} = 0,\ f_{-n}|T_{-2,-1}^{\chi} = f_{-(n+1)};$$

(2) *for $n > 0$.*

$$f_n \mid T_{-2,-1}^{\chi^s} = 0,\ f_n \mid T_{2,3}^{\chi^s} = f_{n+1}.$$

Proof. (1) By definition, for $n \geq 0$,

$$f_{-n}|T_{2,3}^{\chi} = \textstyle\sum_{i\in N_1'/N_3'} i\alpha f_{-n}.$$

As f_{-n} is supported on $K_0\alpha^n I = I\alpha^n I \cup I\beta\alpha^n I$, $\alpha^k i\alpha \in K_0\alpha^n I$ for some $k \geq 0$ and some $i \in N_1'$ implies clearly that $k = n-1$. Therefore $f_0 \mid T_{2,3}^{\chi} = 0$. For $n \geq 1$

$$\begin{aligned}\sum_{i\in N_1'/N_3'} i\alpha f_{-n}(\alpha^{n-1}) &= \sum_{i\in N_1'/N_3'} f_{-n}(\alpha^{n-1}i\alpha)\\ &= \sum_{i\in N_1'/N_3'} \alpha^{n-1}i\alpha^{1-n} f_{-n}(\alpha^n)\\ &= \sum_{i\in N_1'/N_3'} v_0 = 0,\end{aligned}$$

as required.

Also from the definitions,

$$f_{-n}|T^{\chi}_{-2,-1} = \sum_{i\in N_0/N_2} i\alpha^{-1}f_{-n}.$$

As f_{-n} is supported on $K_0\alpha^n I = I\alpha^n I \cup I\beta\alpha^n I$, $\alpha^k i\alpha^{-1} \in I\alpha^n I$ for some $k \geq 0$ and some $i \in N_0$ forces that $k' = n+1$ by Lemma 2.5; for some $k' \geq 0$ and some $i \in N_0$, $\alpha^{k'} i\alpha^{-1} \in I\beta\alpha^n I$ implies clearly that $i \in N_0 \setminus N_2$ and $k' > 0$, and a contradiction is then seen from Lemma 2.5, by applying first Lemma 1.2 to $\alpha i\alpha^{-1}$. Hence,

$$\begin{aligned}\sum_{i\in N_0/N_2} i\alpha^{-1}f_{-n}(\alpha^{n+1}) &= \sum_{i\in N_0/N_2} f_{-n}(\alpha^{n+1}i\alpha^{-1})\\ &= \sum_{i=Id} f_{-n}(\alpha^{n+1}\alpha^{-1}) = v_0,\end{aligned}$$

where we need to check that $\alpha^{n+1}i\alpha^{-1} \notin K_0\alpha^n I$ for $i \in N_0 \setminus N_2$: an application of trick (5) shows that $\alpha^{n+1}i\alpha^{-1} \notin I\alpha^n I$. We are done.

The remaining cases in (2) could be treated in the same way, and we don't give the details. □

Corollary 3.11. *Let σ be an irreducible smooth representations of K_0. Let χ be the character of I on σ^{I_1}. Then, any non-zero $\mathscr{H}(I,\chi)$(resp. $\mathscr{H}(I,\chi^s)$)-submodule of $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$(resp. $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi^s}$) is of finite co-dimension (as a subspace).*

Proof. We verify in detail firstly the regular case $\chi \neq \chi^s$ by using Proposition 3.10.

Let M be a non-zero $\mathscr{H}(I,\chi)$-submodule of $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$. Let ψ be a non-zero vector in M, say $\psi = \sum_{i=-m}^{-n} c_i f_i$, where $-m \leq -n \leq 0$, and $c_{-m}c_{-n} \neq 0$. We could assume further that $n > 0$ by considering the non-zero element $\psi \mid T_{-2,-1}$ (see Proposition 3.10).

Now let M' be the subspace of $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$ generated by M and the set of vectors $\{f_0, f_{-1}, ... f_{-m+1}\}$.

As $c_{-m} \neq 0$, $c_{-m}^{-1}\psi$ minus a linear combination of $f_{-1}, ..., f_{-m+1}$ gives that f_{-m} is in M'. We turn to look at the element $\psi_0 = \psi \mid T_{-2,-1} = \sum_{i=-m}^{-n} c_i f_{i-1}$. Similarly, $c_{-m}^{-1}\psi_0$ minus a linear combination of $f_{-2}, ..., f_{-m}$ gives that f_{-m-1} is in M'. Repeating the former process, we show inductively that all the $f_{-k}, k \geq 0$ is in M'. Hence, $M' = (\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$. We are done in this case.

In the case of $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi^s}$, we note a basis of the former space is $\{f_n, n > 0\}$. In view of (2) of Proposition 3.10, we see the argument that we have just worked out would apply to the current case in the same way.

The degenerate case is a little more complicated, but essentially the same manner as the regular case.

Assume in the following that $\chi = \chi^s$. We will use Proposition 3.9 repeatedly.

Let ψ be a non-zero vector in M. We write it as $\psi = \sum_{i=n}^{m} c_i\psi_i$ where $c_n \cdot c_m \in \tilde{E} \neq 0$ (we allow that $n = m$). We deal with a special case first: $n > 0$.

Now we assume that $n > 0$. Let M' be the subspace of $(\mathrm{ind}_{K_0}^G \sigma)^{I,\chi}$ which is generated by M and the set of vectors $\{\psi_l; l = -(m-1), ..., m-1\}$. Then we will show that $M' = (\mathrm{ind}_{K_0}^G \sigma)^{I,\chi}$ as above. Note that it's not clear that M' is a submodule.

As $c_m \neq 0$, ψ_m is just $c_m^{-1}\psi$ minus a linear combination of the elements ψ_l $(-m+1 \leq l \leq m-1)$ in M', i.e., $\psi_m \in M'$. Now Proposition 3.9 gives $\psi \mid T_{0,-1} = \sum_{i=n}^{m} c_i\psi_{-i} \in M$ (Note that we are in the case all the i are positive .), from which a similar step to that we have just used tells us $\psi_{-m} \in M'$. To proceed, we apply Proposition 3.9 again, we see $\psi \mid T_{0,-1} \cdot T_{2,1} = \sum_{i=n}^{m} c_i\psi_{i+1} \in M$. So ψ_{m+1} is $c_m^{-1}\psi \mid T_{0,-1} \cdot T_{2,1}$ minus a linear combination of the elements ψ_l $(-m \leq l \leq m)$ in M', which means that $\psi_{m+1} \in M'$. Similarly, $\psi \mid T_{0,-1} \cdot T_{2,1} \cdot T_{0,-1} = \sum_{i=n}^{m} c_i\psi_{-(i+1)} \in M$, so by subtracting from it a linear combination of the elements ψ_l $(-m \leq l \leq m)$ we get $\psi_{-(m+1)} \in M'$. We then do induction on the index by considering the right action of $T_{0,-1}$ and $T_{2,1}$ in turn. Then a similar process shows that the generators ψ_{m+k} and $\psi_{-(m+k)}$ are in M' for all $k \geq 0$, i.e., M' contains the basis $\{\psi_n,\ n \in \mathbb{Z}\}$ of $(\mathrm{ind}_{K_0}^G \sigma)^{I,\chi}$.

For the case that n is non-positive, we look at the two elements $\psi' = \psi \mid T_{2,1}, \psi'' = \psi \mid T_{0,-1}$, which are both in M. We claim that they can not be both zero. If $\psi \mid T_{0,-1} = 0$, then $n < 0$, or $n = 0$ and $\psi = c\psi_0$ for some non-zero constant c (we see from Proposition 3.9 the latter case is excluded when $\sigma = \eta \circ \det \otimes St$). The latter case can not happen, as $\psi_0 \mid T_{2,1} = \psi_1 \neq 0$. Assume $n < 0$, we have $n = -m$ and $c_i = c_{-i}$ for $1 \leq i \leq m$. However, in any case, such an element $c_0\psi_0 + \sum_{i=1}^{m} c_i(\psi_i + \psi_{-i})$ $(c_m \neq 0)$ won't become zero under the action of $T_{2,1}$. Hence the claim is true. Now if $\psi' \neq 0$, we apply the argument in the first case $(n > 0)$ to ψ'; otherwise, to the element $\psi'' \mid T_{2,1}$. We are done. $\square$

3.3 The Hecke operator T

In this part, we will calculate the Hecke operator T (c.f. Proposition 3.4) explicitly and explore some applications of the resulting formula.

3.3.1 A first calculation on local systems

Let $(\sigma,\ V)$ be an irreducible smooth representation of K_0. For $g \in G$, $v \in V$, denote by $[g, v]$ the function f in the space of $\mathrm{ind}_{K_0}^G \sigma$, supported on the coset $K_0 g^{-1}$ and satisfying $f(g^{-1}) = v$. We then have the following:

Lemma 3.12. *Let v_0 be a non-zero vector in σ^{I_1}. Then*

$$T\,[Id, v_0] = \textstyle\sum_{u \in N_1/N_2} [u\alpha^{-1}, v_0] + \sum_{u \in N_0/N_2} c_u[\beta u \alpha^{-1}, v_0],$$

where c_u is given by

$$c_u = \begin{cases} \lambda_{\beta,\sigma}, & \text{if } u \in N_1/N_2; \\ \chi_\sigma(\mathrm{diag}(\bar{y}_1, -y_1\bar{y}_1^{-1}, y_1^{-1})), & \text{if } u = n(x_1, y_1) \in (N_0 \backslash N_1)/N_2. \end{cases}$$

Proof. In general, let v be a vector in V, and it's known from (8) of [BL94] that,

$$T\,[Id, v] = \sum_{u \in N_1/N_2} [u\alpha^{-1}, j_\sigma v] + \sum_{u \in N_0/N_2} [\beta u \alpha^{-1}, j_\sigma \sigma(u^{-1}\beta)v]. \tag{12}$$

Let v be a non-zero vector in σ^{I_1}, we then get the first sum, as $j_\sigma(v) = v$.

For the terms with $u \in N_1/N_2$, we know $\sigma(u^{-1})$ acts trivially, as u^{-1} is now in K_0^1. By writing $\sigma(\beta)v$ as the sum of $\lambda_{\beta,\sigma}v$ with some vector in $\sigma(I_1')$, we have shown c_u is the case, when $u \in N_1/N_2$.

For the terms with $u \in (N_0 \backslash N_1)/N_2$, one needs some calculations to simplify the vector $j_\sigma \cdot \sigma(u^{-1}\beta)v$. We write u^{-1} as $n(x_1, y_1)$, and the condition on u implies that $y_1 \in U_E$, and then Lemma 1.2 gives

$$u^{-1}\beta = \beta n(\bar{y}_1^{-1}x_1, y_1^{-1})\beta \cdot \mathrm{diag}(y_1, -\bar{y}_1 y_1^{-1}, \bar{y}_1^{-1}) \cdot n_u,$$

where n_u is some element in $N_0(\subset I_1)$. Put $n_u' = \beta n(\bar{y}_1^{-1}x_1, y_1^{-1})\beta$, which is an element in $N_0'(\subset I_1')$.

Hence,

$$\begin{aligned} j_\sigma \cdot \sigma(u^{-1}\beta)v &= j_\sigma \cdot \sigma(n_u' \cdot \mathrm{diag}(y_1, -\bar{y}_1 y_1^{-1}, \bar{y}_1^{-1}))v \\ &= \chi_\sigma(\mathrm{diag}(y_1, -\bar{y}_1 y_1^{-1}, \bar{y}_1^{-1})) j_\sigma \cdot \sigma(n_u')v \\ &= \chi_\sigma(\mathrm{diag}(y_1, -\bar{y}_1 y_1^{-1}, \bar{y}_1^{-1})) \cdot v, \end{aligned}$$

where the last equality holds as $\sigma(n_u')v - v \in \sigma(I_1')$. We are done. $\square$

3.3.2 A second calculation on local systems

In the case that σ is one-dimensional, we have a formula for the Hecke operator T, Lemma 3.12. For later applications, we would like to make it more explicit.

From a recent result of Henniart-Vignéras ([HV12], Proposition 3.17), we know $\lambda_{\beta,\sigma} = 0$ for any irreducible representations σ with dimension bigger than 1, as β is not in $B'B$. In other words, this means $\beta \cdot v_0$ lies in $\sigma(I'_1)$. This is our start point.

Let σ be an irreducible representation of K_0. From now on, we assume $\dim \sigma = r > 1$. Also, we know $r \leq q^3$. Fix a non-zero vector v_0 in σ^{I_1}. Therefore, from what we have just described, there are r elements $\{u_i,\ 1 \leq i \leq r\}$ in N'_0/N'_1 such that

$$N'_\sigma = \{n_i \cdot v_0,\ 1 \leq i \leq r\} \text{ is a basis for the space } W \text{ of } \sigma.$$

Furthermore, we always assume that v_0 is in the above set.

To state the following Proposition, we need to introduce some notations. Denote by S_{v_0} the following linear functional on W:

$$\begin{aligned} S_{v_0} :\ W &\rightarrow \tilde{E} \\ v\ &\mapsto \textstyle\sum_{n \in N'_0/N'_1} l_n(v), \end{aligned}$$

where we write $v = \sum_{n_i \in N'_0/N'_1} l_{n_i}(v) n_i \cdot v_0$ and put $l_n(v) = 0$ for n outside the finite set $\{n_i\}$ chosen above. As before, let $\chi_\sigma = \chi_1 \otimes \chi_2$ be the character of I acting on the σ^{I_1}. Then:

Proposition 3.13. *For a vector $v \in W$,*

$$T\left([Id, v]\right) = \sum_{u \in N_1/N_2} [u\alpha^{-1},\ S_{v_0}(v) v_0] + \sum_{u \in N_0/N_2} [\beta u \alpha^{-1},\ L_{v_0,u}(v) v_0],$$

where $L_{v_0,u}$ is the linear functional on W defined by

$$L_{v_0,u}(v) = \textstyle\sum_{n \in N'_0/N'_1} c_{n_s^{-1}u} \cdot l_n(v),$$

where we denote $\beta n \beta$ by n_s.

Proof. It is just direct calculation. We insert $v = \sum_{n \in N'_0/N'_1} l_n(v) n \cdot v_0$ into the formula $T([Id, v])$. Combining with Lemma 3.12, we see

$$\begin{aligned} T([Id, v]) &= \sum_{n \in N'_0/N'_1} l_n(v) n \cdot T([Id, v_0]) \\ &= \sum_{n \in N'_0/N'_1} l_n(v) \left(\sum_{u \in N_1/N_2} [nu\alpha^{-1},\ v_0] + \sum_{u \in (N_0 \backslash N_1)/N_2} c_u [n\beta u\alpha^{-1}, v_0] \right). \end{aligned}$$

We begin to simplify each term in the above sum, according to their nature.

For the term $u \in N_2$, we see $\alpha u \alpha^{-1} \in N_0$ and $\alpha n \alpha^{-1} \in N_1'$.

For the terms $u \in (N_1 \setminus N_2)/N_2$, which we write as $n(0, \varpi_E y_1)$, an application of Lemma 1.2 to βu gives

$$n u \alpha^{-1} = \beta n_s \beta u \alpha^{-1} = \beta n(0, \varpi_E^{-1} y_1^{-1}) n_s l h,$$

where l is some element in N_1', and h is the diagonal matrix

$$\begin{pmatrix} -y_1^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y_1 \end{pmatrix}.$$

Hence, noting $n_s l h \in K_0$, and applying Lemma 1.2 again to $\beta n(0, \varpi_E^{-1} y_1^{-1})$, we came to the first sum in the Proposition.

Before dealing with the remaining terms where u goes through $(N_0 \setminus N_1)/N_2$, we note firstly that the constant c_u indeed depends only on the class uN_1. We will use this without comment in the following. Secondly, as $c_u = 0$ for $u \in N_1/N_2$, the corresponding part resulting from the original formula $T([Id,\ v_0])$ vanishes.

We now exchange the order of the sum:

$$\sum_{u \in (N_0 \setminus N_1)/N_2} \textstyle\sum_{n \in N_0'/N_1'} [n \beta u \alpha^{-1},\ c_u l_n(v) v_0].$$

For $n \in N_1'$, we get

$$\sum_{u \in (N_0 \setminus N_1)/N_2} [\beta u \alpha^{-1},\ c_u l_{Id}(v) v_0].$$

For $n \in (N_0' \setminus N_1')/N_1'$, we split u as $u_1 u_2$, where $u_1 \in (N_0 \setminus N_1)/N_1$ and $u_2 \in N_1/N_2$. Then we decompose the remaining part of the above sum as

$$\sum_{n \in (N_0' \setminus N_1')/N_1'} \textstyle\sum_{u_2 \in N_1/N_2} (\sum_{u_1 \neq n_s^{-1}} + \text{remaining term}).$$

In the above, the sum over $u_1 = n_s^{-1}$ gives us directly the part in the sum of Proposition where u goes through N_1/N_2:

$$\textstyle\sum_{u \in N_1/N_2} \left[\beta u \alpha^{-1},\ \left(\displaystyle\sum_{n \in (N_0' \setminus N_1')/N_1'} c_{n_s^{-1}} l_n(v)\right) v_0\right].$$

For the part in which $u_1 \neq n_s^{-1}$, we note that $u_3 := n_s n_1$ goes through $(N_0 \setminus N_1)/N_1 - n_s N_1$, when u_1 goes through $(N_0 \setminus N_1)/N_1 - n_s^{-1}N_1$. We note that $c_{Id} = 0$. Therefore, we get

$$\sum_{n\in(N_0'\setminus N_1')/N_1'} l_n(v) \sum_{u_2\in N_1/N_2} \sum_{u_3\in(N_0\setminus N_1)/N_1} [\beta u_3 u_2 \alpha^{-1},\ c_{n_s^{-1}u_3} v_0]$$
$$= \sum_{u_3\in(N_0\setminus N_1)/N_1} \sum_{u_2\in N_1/N_2} \left[\beta u_3 u_2\alpha^{-1},\ \left(\sum_{n\in(N_0'\setminus N_1')/N_1'} c_{n_s^{-1}u_3} l_n(v)\right) v_0\right]$$
$$= \sum_{u\in(N_0\setminus N_1)/N_2} \left[\beta u\alpha^{-1},\ \left(\sum_{n\in(N_0'\setminus N_1')/N_1'} c_{n_s^{-1}u} l_n(v)\right) v_0\right].$$

Finally, we put all the terms together and the formula comes out. □

Remark 3.14. *As the choice of v_0 is only up to a scalar, we see $S_{v_0}(v)v_0$ and $L_{v_0,u}(v)v_0$ only depend on the vector v. We also note that the functional $L_{v_0,u}$ only depends on the residue class uN_1.*

We already know from the definition that j_σ is close to the projection from σ to σ^{I_1}, and the following corollary makes this precise:

Corollary 3.15. *Let (σ, W) be an irreducible smooth representation of K_0 such that $\dim\sigma > 1$, and v_0 be a non-zero vector in σ^{I_1}. For any vector $v \in W$, we have the following*

$$j_\sigma v = S_{v_0}(v)v_0,\ \ j_\sigma\sigma(u^{-1}\beta)v = L_{v_0,u}(v)v_0,\ \textit{for}\ u \in N_0/N_2.$$

In the case that $\sigma = St$, we have a simplified version of the Hecke operator.

Corollary 3.16. *Let σ be the inflation of St, we then have*

$$T([Id, v]) = \sum_{u\in N_1/N_2} [u\alpha^{-1},\ S_{v_0}(v)v_0] + \sum_{u\in N_0/N_2} [\beta u\alpha^{-1},\ (S_{v_0}(v) - l_{y_s}(v))v_0].$$

Proof. In this case, $\chi_\sigma = 1$, $c_{Id} = 0$. □

3.4 Is $\mathrm{ind}_{K_0}^G\sigma$ free over $\mathscr{H}(K_0,\sigma)$?

In this section, we pursue an application of the Hecke operator formula T. It seems reasonable to propose the following:

Conjecture 3.17. *For an irreducible smooth representation σ of K_0, the compactly induced representation $\mathrm{ind}_{K_0}^G\sigma$ is a free module of infinite rank over the algebra $\mathscr{H}(K_0,\sigma)$.*

However, we have the following weaker result in general.

Proposition 3.18. *For an irreducible smooth representation σ of K_0, the compact induced representation $\mathrm{ind}_{K_0}^G\sigma$ is faithfully flat over the algebra $\mathscr{H}(K_0,\sigma)$.*

Proof. Recall we have shown $\mathscr{H}(K_0,\sigma)$ is a polynomial algebra in one variable over $\tilde{E}$, especially it is a Dedekind domain. As it is well-known that flatness is equivalent to torsion-free over a Dedekind domain, we will be done if the latter point is checked in our case.

In her thesis [Abd11], Abdellatif has shown that T is injective (Théorème 4.5.14), which in particular implies that $\mathrm{ind}_{K_0}^G\sigma$ is torsion-free over $\mathscr{H}(K_0,\sigma)$. The result follows. □

Remark 3.19. *Florian Herzig [Her11] has proved maximal compact induction is torsion-free over the corresponding spherical Hecke algebra, when G is F-split.*

We start with some general setting, and then prove some special cases of 3.17.

For $n\geq 0$, denote by $B_{n,\sigma}$ the set of sections in $\mathrm{ind}_{K_0}^G\sigma$ which are supported in the ball of the tree of radius $2n$ around the vertex $\mathbf{v}_0$. Let $C_{n,\sigma}$ be the set of sections in $\mathrm{ind}_{K_0}^G\sigma$ which are supported in the circle of radius $2n$ around the vertex $\mathbf{v}_0$.

Assumption 3.20. *Let $f\in B_{n+1,\sigma}$. If $Tf\in B_{n+1,\sigma}$, then $f\in B_{n,\sigma}$.*

We note that, after a simple consideration on the tree, the Assumption 3.20 is equivalent to the statement that: *In the following q^4+q linear functional, say, q copies of $j_\sigma v$ and the q^4 linear functional $j_\sigma\sigma(u^{-1}\beta)v$ for all $u\in N_0/N_2$, any q^4 vanishing of them implies the vanishing of v.*

Lemma 3.21. *The Assumption 3.20 is true in the cases that $\sigma=\eta\circ det$ or $\eta\circ det\otimes St$ for any character η of k_E^1.*

Proof. The statement prior to the Lemma is trivial when σ is a one-dimensional character.

Next, we consider the case that $\sigma=\eta\circ\det\otimes St$. After a twist, we are reduced to consider $\sigma=St$. It is pleasant to verify the above statement in this case, using Corollary 3.16. Assume firstly, for all the $u\in N_0/N_2$, $S_{v_0}(v)-l_{u_s}(v)=0$. Adding these equations together, we see immediately that $S_{v_0}(v)=0$. Hence, all the $l_{u_s}(v)$ are zero. Therefore, we only need to consider the case of $S_{v_0}(v)$ and any q^3-1 of the $S_{v_0}(v)-l_{u_s}(v)$ being zero. This forces all $l_{u_s}(v)$ being vanishing clearly. We are done in this case. □

Proposition 3.22. *Assumption 3.20 implies Conjecture 3.17.*

Proof. Using Assumption 3.20, by induction we can follow [BL94] to find a subset A_n of $C_{n,\sigma}$, satisfying that $\sqcup_{2k+2i\leq 2n} T^i A_k$ forms a basis of $B_{n,\sigma}$.

For $n = 0$, take $A_0 = \{[Id, u_j]\}$, where $\{u_j\}$ is a basis of the underlying space of σ. Assume the former statement is done for n. Then we need to show the set $\sqcup_{\substack{2k+2i\leq 2n+2,\\ k\leq n}} T^i A_k$ is linearly independent.

Assume the claim is false and we have a linear combination of elements from $\sqcup_{\substack{2k+2i\leq 2n+2,\\ k\leq n}} T^i A_k$. As $\sqcup_{\substack{2k+2i\leq 2n+2,\\ k\leq n}} T^i A_k$ is the union of $\sqcup_{\substack{k+i=n+1,\\ k\leq n}} T^i A_k$ and $\sqcup_{k+i\leq n} T^i A_k$, we get an element f, lying in the ball $B_{n,\sigma}$, and also $Tf \in B_{n,\sigma}$. Now Assumption 3.20 ensures that $f \in B_{n-1,\sigma}$. This means that the projection of f to the circle of radius $2n$ around the vertex v_0 is zero. We recall that f is a linear combination of elements from $\sqcup_{k+i=n} T^i A_k$ and the projection of any non-zero element in $\sqcup_{k+i=n} T^i A_k$ is non-zero. The induction hypothesis for n already implies that the projection of $\sqcup_{k+i=n} T^i A_k$ is a basis for $C_{n,\sigma}$, hence the former statement forces the vanishing of f. We are done for the claim in the last paragraph. We then proceed to choose a subset A_{n+1} of the form $\{[g,\ u]\}_{g,u}$, supported in the circle of radius $2n+2$, and complete $\sqcup_{\substack{2k+2i\leq 2n+2,\\ k\leq n}} T^i A_k$ to a basis of $B_{n+1,\sigma}$. This is possible, and we only need to complete the projection of $\sqcup_{\substack{k+i=n+1,\\ k\leq n}} T^i A_k$ to a basis of $C_{n+1,\sigma}$.

In summary, we have chosen a family of $A_n \subset C_{n,\sigma}$ satisfying $\cup_{n\geq 0} \sqcup_{2k+2i\leq 2n} T^i A_k$ is basis of the compact induction $\mathrm{ind}_{K_0}^G \sigma$. In particular, the set $\cup_{n\geq 0} A_n$ is a basis of $\mathrm{ind}_{K_0}^G \sigma$ over $\mathscr{H}(K_0, \sigma)$. □

3.5 The right action of $\mathscr{H}(K_0, \sigma)$ on the (K_0, σ)-isotypic component of principal series representations

Lemma 3.23. *For a character ε of B and an irreducible smooth representation σ of K_0, the space $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma,\ \mathrm{ind}_B^G\, \varepsilon)$ is at most one-dimensional, and it is non-zero if and only if*

$$\varepsilon_0 = \chi_\sigma^s,$$

where ε_0 is the restriction of ε to H_0.

Proof. We have:

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_B^G\,\varepsilon) &\cong \mathrm{Hom}_{K_0}\,(\sigma,\ \mathrm{ind}_B^G\varepsilon|_{K_0})\\ &\cong \mathrm{Hom}_{K_0}\,(\sigma,\ \mathrm{ind}_{B\cap K_0}^{K_0}\varepsilon)\\ &\cong \mathrm{Hom}_{K_0}\,(\sigma,\ (\mathrm{ind}_{B\cap K_0}^{K_0}\varepsilon)^{K_0^1})\\ &\cong \mathrm{Hom}_{\overline{G}}\,(\sigma,\ \mathrm{ind}_{B_0}^{\Gamma_0}\varepsilon_0)\\ &\cong \mathrm{Hom}_{B_0}\,(\sigma|_{B_0},\varepsilon_0).\end{aligned}$$

The first and the last isomorphism are from Frobenius reciprocity. The second is by the decomposition $G = BK_0$. The third is true because the group K_0^1 acts trivially in the irreducible representation σ. As $B\cap K_0/B\cap K_0^1\cong B_0$, the character ε is a lift of ε_0, via this isomorphism. Hence, we can identify $(\mathrm{ind}_{B\cap K_0}^{K_0}\varepsilon)^{K_0^1}$ with $\mathrm{ind}_{B_0}^{\overline{G}}\varepsilon_0$ (as representations of $\overline{G}(k_F)$), which gives the fourth isomorphism.

We proceed to deal with the last space, and we see

$$\mathrm{Hom}_{B_0}\,(\sigma|_{B_0},\varepsilon_0)\cong\ \mathrm{Hom}_{U_0}\,(\sigma|_{B_0},\varepsilon_0)^{B_0/U_0}.$$

Put $v_0' = \beta v_0 \in \sigma^{U_0'}$. Lemma 3.1 implies that v_0' generates the space σ_{U_0}. Let $l_{v_0'}$ be the U_0-map

$$\begin{aligned} l_{v_0'}:\ \sigma_{U_0} &\to \tilde{E}\\ v_0' &\mapsto 1.\end{aligned}$$

Then $l_{v_0'}$ generates the space $\mathrm{Hom}_{U_0}\,(\sigma|_{B_0},\varepsilon_0)$.

Hence $\mathrm{Hom}_{U_0}\,(\sigma|_{B_0},\varepsilon_0)^{B_0/U_0}\neq 0$, if and only if

$$l_{v_0'} \text{ is } B_0/U_0\text{-invariant}.$$

But this is just

$$\chi_\sigma^s = \varepsilon_0,$$

and we are done. □

We would like to specify a non-zero G-morphism in the above argument, relative to v_0. Let $g\in G$, and we write it as bk, where $b\in B,\ k\in K_0$. Define $P_{v_0',0}$ as the map in $\mathrm{Hom}_{K_0}\,(\sigma,\ \mathrm{ind}_B^G\varepsilon)$:

$$P_{v_0',0}(v)(g) =\ \varepsilon(b)l_{v_0'}(\bar{k}v),$$

where $v\in V$, $\bar{k}$ is the image of k in $\overline{G}(k_F)$. Put $\mathfrak{f}_0 = P_{v_0',0}(v_0)$. We note that $P_{v_0',0}$ is well-defined, by the definition of $l_{v_0'}$. Then, by Frobenius reciprocity, we get a map $P_{v_0',1}\in\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_B^G\,\varepsilon)$, which corresponds to $P_{v_0',0}$, determined by the condition that

$$P_{v'_0,1}([Id,v]) = P_{v'_0,0}(v), \text{ for any } v\in V.$$

We determine explicitly the constant c, such that $P_{v'_0,1}|T = c\cdot P_{v'_0,1}$

Proposition 3.24. *The G-morphism $P_{v'_0,1}$ generates the space* $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma, \mathrm{ind}_B^G\,\varepsilon)$, *and the operator T in Corollary 3.4 acts as the scalar c_ε on it, i.e.,*

$$P_{v'_0,1}|T = c_\varepsilon\cdot P_{v'_0,1},$$

where c_ε is

$$c_\varepsilon = \varepsilon(\alpha)+\textstyle\sum_{y_1\in k_E^\times;\ y_1+\bar{y}_1=0}\ \varepsilon_0(-y_1^{-1},1,y_1).$$

Proof. We verify by definition that $\mathfrak{f}_0(Id)=\lambda_{\beta,\sigma}$, $\mathfrak{f}_0(\beta)=1$. Hence, $c_\varepsilon = P_{v'_0,1}|T([Id,v_0])(\beta)$. As $P_{v'_0,1}|T([Id,v_0])$ is just $P_{v'_0,1}(T([Id,v_0]))$, for which we can use Proposition 3.12 to calculate explicitly.

We compute the first partial sum, say:

$$\textstyle\sum_{u\in N_1/N_2}\ \mathfrak{f}_0(\beta u\alpha^{-1}).$$

The term with $u\in N_2$ contributes $\varepsilon(\alpha)$, as $\mathfrak{f}_0(\beta)=1$. For the remaining terms $u\in(N_1\setminus N_2)/N_2$, we write u as $n(0,\varpi_E y_1)$, where y_1 goes through the set $L_1^* = L_1\setminus\{0\}$. An application of Lemma 1.2 gives that:

$$\beta u\alpha^{-1} = n(0,\varpi_E^{-1}y_1^{-1})\mathrm{diag}(-y_1^{-1},1,y_1)n'(0,\varpi_E y_1^{-1}).$$

Note that $\mathfrak{f}_0$ is I_1-invariant, as v_0 is. In summary, we get

$$\textstyle\sum_{u\in N_1/N_2}\ \mathfrak{f}_0(\beta u\alpha^{-1}) = \varepsilon(\alpha)+\sum_{y_1\in L_1^*}\ \varepsilon_0(-y_1^{-1},1,y_1).$$

For the second partial sum, it is immediate to see,

$$\textstyle\lambda_{\beta,\sigma}\sum_{u\in N_1/N_2}\ \mathfrak{f}_0(u\alpha^{-1}) = \lambda_{\beta,\sigma}\cdot\varepsilon(\alpha^{-1})\sum_{u\in N_1/N_2}\ \mathfrak{f}_0(Id)=0.$$

For the last partial sum, with $u\in(N_0\setminus N_1)/N_2$, we simplify it as

$$\textstyle\sum_{u\in(N_0\setminus N_1)/N_2}\ c_u\cdot\mathfrak{f}_0(u\alpha^{-1}) = \lambda_{\beta,\sigma}\cdot\varepsilon(\alpha^{-1})\sum_{u\in(N_0\setminus N_1)/N_2}\ c_u.$$

Hence, we are reduced to compute

$$\textstyle\sum_{u\in(N_0\setminus N_1)/N_2}\ \chi_\sigma(\mathrm{diag}(y_1,-\bar{y}_1y_1^{-1},\bar{y}_1^{-1})),$$

where $u^{-1}=n(x_1,y_1)$. But it vanishes in any case, as one can check as follows:

Write ε_0 as $\chi_1\otimes\chi_2$ and hence we need to determine

$$\textstyle\sum_{u\in(N_0\setminus N_1)/N_2}\ \chi_1(\bar{y}_1^{-1})\chi_2(-\bar{y}_1y_1^{-1}),$$

where we have used Lemma 3.23.

We write furthermore $u=n(x_1,y_1+\varpi_E y_2)$, where $y_2\in L_1$, and $(x_1,y_1)\in L_2$ with $y_1\in\mathfrak{o}_E^\times$. Hence, the above sum is changed into

$\sum_{y_2\in L_1}\sum_{(x_1,y_1)\in L_{2,\mathfrak{p}_E}} \chi_1((y_1+\varpi_E y_2)^{-1})\chi_2(-(y_1+\varpi_E y_2)(\bar{y}_1+\varpi_E\bar{y}_2)^{-1}),$

where $L_{2,\mathfrak{p}_E}$ is the subset of L_2 with elements $n(x,y)$ satisfying $y\in\mathfrak{o}_E^\times$.

As the character χ_σ is defined from the action of I on the line σ^{I_1}, χ_1 is trivial on U_E^1 and χ_2 is trivial on $E^1\cap U_E^1$; hence we get

$$\sum_{y_2\in L_1}\sum_{(x_1,y_1)\in L_{2,\mathfrak{p}_E}} \chi_1(y_1^{-1})\chi_2(-y_1\bar{y}_1^{-1})=0.$$

We have shown the claim. □

The constant c_ε above is explicit in the following sense:

Corollary 3.25. *We determine c_ε as:*

$$c_\varepsilon=\begin{cases}\varepsilon(\alpha)-\chi_1'(-1), & \textit{if}\ \varepsilon_0=\varepsilon_0^s;\\ \varepsilon(\alpha), & \textit{otherwise}.\end{cases}$$

Proof. We have already done this calculation in detail, say the argument of Proposition 2.6, and Proposition 2.31. □

3.6 The Bruhat-Tits tree of G

3.6.1 Height and antecedent

Recall we have fixed a standard apartment $\{\mathbf{v}_k, k\in\mathbb{Z}\}$. Denote by ∞ the positive end of this standard apartment. For any vertex $\mathbf{v}$, let $\overline{\mathbf{v}\infty}$ be the geodesic ray (i.e., the unique path between $\mathbf{v}$ and ∞) from $\mathbf{v}$ to ∞. So we can find an integer k such that $\mathbf{v}_k\in\overline{\mathbf{v}\infty}$. Define the height $h(\mathbf{v})$ of $\mathbf{v}$ as $k-d(\mathbf{v}_k,\mathbf{v})$. Note that this definition is independent of the choice of k and that $h(\mathbf{v}_k)=k$.

Given any two vertices $\mathbf{v}$ and $\mathbf{v}'$, *we say* $\mathbf{v}$ *is under* $\mathbf{v}'$, *if* $\mathbf{v}'\in\overline{\mathbf{v}\infty}$. The following two lemmas will be used later:

Lemma 3.26. $(N/N_{-r})\mathbf{v}_r=\{\mathbf{v}\in X_0:\ h(\mathbf{v})=r\}$

Proof. Firstly, we note that the stabilizers of $\mathbf{v}_{2k}$ and $\mathbf{v}_{2k+1}$ in G are respectively $\alpha^k K_0\alpha^{-k}$ and $\alpha^k K_1\alpha^{-k}$ for any integer k. Therefore the stabilizers of $\mathbf{v}_{2k}$ and $\mathbf{v}_{2k+1}$ in N are respectively $N\cap\alpha^k K_0\alpha^{-k}$ and $N\cap\alpha^k K_1\alpha^{-k}$. But these are exactly N_{-2k} and $N_{-(2k+1)}$.

Secondly, we are going to show: for a non-negative integer l and an integer r, $(N_{-(r+l)}/N_{-r})\mathbf{v}_r=\{\mathbf{v}\in X_0:\ h(\mathbf{v})=r,\ \mathbf{v}_{r+l}\in\overline{\mathbf{v}\infty}\}$. There are two steps:

Step 1 For $u\in N$, $u\mathbf{v}_r$ is also of height r. Take an integer k such that u fixes $\mathbf{v}_k$ and $\mathbf{v}_k\in\overline{u\mathbf{v}_r\infty}$. So by definition, $h(u\mathbf{v}_r)=k-d(\mathbf{v}_k,u\mathbf{v}_r)$, which equals r by the choice of k. As we have fixed ∞, we understand that

$u\mathbf{v}_{r+l} \in \overline{u\mathbf{v}_r\infty}$ for all non-negative integers l. In particular, when we restrict to $u \in N_{-(r+l)}$, we get $\mathbf{v}_{r+l} \in \overline{u\mathbf{v}_r\infty}$. We have shown that $(N_{-(r+l)}/N_{-r})\mathbf{v}_r$ is contained in $\{\mathbf{v} \in X_0 :\ h(\mathbf{v}) = r,\ \mathbf{v}_{r+l} \in \overline{\mathbf{v}\infty}\}$.

Step 2 We finish this step by counting. For a non-negative integer l and an integer r, denote respectively by n_r^l and m_r^l the cardinality of $N_{-(r+l)}/N_{-r}$ and that of the set M_r^l, where M_r^l is $\{\mathbf{v} \in X_0 :\ h(\mathbf{v}) = r,\ \mathbf{v}_{r+l} \in \overline{\mathbf{v}\infty}\}$. The list for n_r^l is as follows:

$$n_r^l \ = \begin{cases} q^{2l}, & \text{if } l \text{ is even,} \\ q^{2l+(-1)^{r-1}}, & \text{if } l \text{ is odd.} \end{cases} \tag{13}$$

To see this, we reduce the above to two special cases by conjugating by some power of α: n_{-l}^l and n_{1-l}^l, namely the cardinality of N_0/N_l and N_{-1}/N_{l-1}.

We deal with n_{-l}^l in detail. Given an even l, we have $n_{-l}^l = (n_{-2}^2)^{\frac{l}{2}}$. But $n_{-2}^2 = n_{-1}^1 \cdot n_{-2}^1 = q^3 \cdot q = q^4$. So in this case $n_{-l}^l = q^{2l}$. When l is odd, $n_{-l}^1 = n_{-1}^1 = q^3$. Now $l-1$ is even, and from the even case we get $n_{-l}^l = q^{2(l-1)} \cdot q^3 = q^{2l+1}$. Similarly we can show n_{1-l}^l as required in (13).

To compute m_r^l, we firstly note that there exists an induction relation between them by observing the tree: $m_r^{l+1} = m_r^l \cdot c_{r+l+1}$, where we denote by c_t the number of vertices adjacent to and under v_t for any integer t. We know that it equals q or q^3, depending on whether t is odd or not. So we only need to compute some initial cases. The result is: $m_r^0 = 1$ for any r, $m_r^1 = q$ or q^3, depending on whether r is even or not. Combining the initial cases and the induction relation, we have finally shown that m_r^l is exactly given by the formula in (13).

We have finished the proof of the Lemma. ∎

Definition 3.27. *For a vertex $\mathbf{v} \in X_0$ and a positive integer n, the n-*antecedent *$a^n(\mathbf{v})$ of $\mathbf{v}$ is the unique vertex of height $h(\mathbf{v}) + n$ which is of distance n from $\mathbf{v}$.*

Remark 3.28. *The definition above is well-defined because there exists a unique path from $\mathbf{v}$ to ∞. From that we naturally have $\overline{\mathbf{v}\infty} = (\mathbf{v}, ..., a^n(\mathbf{v}), ...)$ for any vertex $\mathbf{v} \in X_0$.*

Lemma 3.29. *$a^l(u\mathbf{v}_k) = u\mathbf{v}_{k+l}$ for all positive integers l and all $k \in \mathbb{Z}$, and all $u \in N$.*

Proof. The l-antecedent of $\mathbf{v}_k$ is $\mathbf{v}_{k+l}$ by definition above. As the action of N preserves height (as we have already observed in the proof of Lemma 3.26) and distance, we are done. ∎

Remark 3.30. *Formally, $a^l(g\cdot\mathbf{v}) = g\cdot a^l(\mathbf{v})$ holds for any $l\geq 0$ and $g\in G$, and any $\mathbf{v}\in X_0$.*

In view of Proposition 3.13, we could generalize the 2-antecedent as follows

Definition 3.31. $A[n_0\alpha^k, v] = \begin{cases} [n_0\alpha^{k+1}, \eta(-1)v], & \text{if } \sigma = \eta\circ det, \\ [n_0\alpha^{k+1}, L_{v_0,Id}(v)v_0'], & \text{otherwise.}\end{cases}$

Remark 3.32. *One can check directly from the definition of $L_{v_0,Id}$ that $A[n_0\alpha^k, v]$ is independent of the choice of v_0.*

3.6.2 A rough estimation of I_1-actions on the tree

Recall again we have fixed a standard apartment on the tree of G, say $\{\mathbf{v}_k,\ k\in\mathbb{Z}\}$. For any vertex $\mathbf{v}$, we have mentioned before that $\mathbf{v}$ is under some $\mathbf{v}_k$, for some integer k, i.e., there is a vertex $\mathbf{v}_k$ such that $\mathbf{v}_k\in\overline{\mathbf{v}\infty}$. As a result, one can associate any vertex $\mathbf{v}$ a unique integer $n_{\mathbf{v}}$ which is the least integer satisfying the former property. In our former notation, we have clearly

$$n_{\mathbf{v}} = h(\mathbf{v}) + d(\mathbf{v}_{n_{\mathbf{v}}}, \mathbf{v})$$

Following is the main property known to us about $n_{\mathbf{v}}$ and a rough estimation of the distance between $\mathbf{v}$ and $u'\cdot\mathbf{v}$, for a $u'\in N_1'$.

Proposition 3.33. *Let $\mathbf{v}$ be a vertex under $\mathbf{v}_0$, such that $d(\mathbf{v},\mathbf{v}_0) = 2r(>0)$. Then we have for $u'\in N_1'$,*

$$n_{u'\cdot\mathbf{v}} = n_{\mathbf{v}} \text{ and } d(u'\cdot\mathbf{v},\mathbf{v}) < 2(2r+n_{\mathbf{v}})$$

where $2r+n_{\mathbf{v}}$ is the distance from $\mathbf{v}$ to $\mathbf{v}_{n_{\mathbf{v}}}$.

Proof. The proof is simple matrix calculation. □

Remark 3.34. *It seems to the author not much could be said beyond the inequality of the last Proposition.*

3.7 The image of $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$ under the Hecke operator T

In subsection 3.2, we have investigated the I_1-invariants of $\mathrm{ind}_{K_0}^G\sigma$. As the Hecke operator T respects the group action, it is reasonable to know how $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$ behaves under the map T. In the following proposition, we re-write the basis $\{f_n,\ n\in\mathbb{Z}\}$ in terms of local systems.

Proposition 3.35. *With the same notations as before, we have*

$$f_n = \begin{cases} \sum_{i \in N_0/N_{2m}} [i\alpha^{-m},\ v_0], & n = -m \le 0 \\ \sum_{j \in N_1'/N_{2n}'} [j\alpha^n, v_0'], & n > 0 \end{cases}$$

Proof. Directly from the definitions of f_n in last subsection. □

Definition 3.36. $R_n^+(\sigma) = [N_0\alpha^{-n}, \sigma]$, $n \ge 0$; $R_{n-1}^-(\sigma) = [N_1'\alpha^n, \sigma]$, $n \ge 1$

We also put $R_0(\sigma) = R_0^+(\sigma) = R_{-1}^-(\sigma)$.

We have an initial estimation then:

Proposition 3.37. (1).

$$T(R_0(\sigma)) \subseteq R_1^+(\sigma) \oplus R_0^-(\sigma),$$
$$T(R_n^+(\sigma)) \subseteq R_{n-1}^+(\sigma) \oplus R_n^+(\sigma) \oplus R_{n+1}^+(\sigma), n \ge 1.$$

$$(2).\ T(R_n^-(\sigma)) \subseteq R_{n-1}^-(\sigma) \oplus R_n^-(\sigma) \oplus R_{n+1}^-(\sigma), n \ge 0.$$

Proof. Actually, this Proposition could be seen from the tree of G if one keeps the action of T in mind.

The first inclusion in (1) follows directly from (12) and our definition. For the second inclusion, it is clear that $\alpha^{-n}u\alpha^{-1} \in N_0\alpha^{-(n+1)}$ for $u \in N_1/N_2$, as $n \ge 1$. Then we check the following, which completes the argument of (1):

$$\alpha^{-n}\beta u\alpha^{-1} = \begin{cases} \alpha^{1-n}(\beta\alpha y\alpha^{-1}), & \text{if } u \in N_2, \\ n(0, \varpi_E^{2n-1}y_1^{-1})\alpha^{-n}i_1, & \text{if } u \in N_1 \setminus N_2, \\ n_1\alpha^{-(n+1)} \cdot i_2, & u \in N_0 \setminus N_1, \end{cases}$$

for some $i_1,\ i_2 \in I$ and $n_1 \in N_{2n}$.

For (2), let n be a non-negative integer. At first, it is easy to see $\alpha^{n+1} \cdot \beta u\alpha^{-1} \in N_1'\alpha^{n+2}\beta$. Then we check the following after some calculations, which finishes the proof of (2):

$$N_1'\alpha^{n+1}u\alpha^{-1} = \begin{cases} N_1'\alpha^n \cdot (\alpha u\alpha^{-1}), & \text{if } u \in N_2, \\ N_1'\alpha^{n+1}\beta i_3, & \text{if } u \in N_1 \setminus N_2, \end{cases}$$

for some $i_3 \in I$. □

The following result is a refinement of Proposition 3.7, and it will play a role later.

Lemma 3.38. (1). *For $n \ge 0$, the N_0-invariants of the space $R_n^+(\sigma)$ is one-dimensional and generated by f_{-n}.*

(2). *For $n \ge 1$, the N_1'-invariants of the space $R_{n-1}^-(\sigma)$ is one-dimensional and generated by f_n.*

Proof. (1). Firstly, we note that $K_0\alpha^n I_1 = K_0\alpha^n N_0$. Hence, a non-zero function f in $R_n^+(\sigma)^{N_0}$ would indeed have support on $K_0\alpha^n I_1$. We need to look at $f(\alpha^n i)$ for $i \in N_0$, which is indeed $f(\alpha^n)$ as f is fixed by N_0. However, being a vector in the underlying space of σ, it is fixed by the lower-triangular subgroup of I_1. Therefore, $f(\alpha^n) \in \sigma^{I_1}$. Then again the condition f is fixed by N_0 will force f to differ from f_{-n} only by a scalar. We are done.

The proof of (2) is similar, and we omit the details. □

From Proposition 3.37, we see $T\mid_{R_n^+(\sigma)}$ is the sum of I-morphisms $T^- : R_n^+(\sigma) \to R_{n-1}^+(\sigma)$ and $T^+ : R_n^+(\sigma) \to R_n^+(\sigma) \oplus R_{n+1}^+(\sigma)$, for $n \geq 1$. Similarly, $T\mid_{R_n^-(\sigma)}$ is the sum of I-morphisms $T^- : R_n^-(\sigma) \to R_{n-1}^-(\sigma)$ and $T^+ : R_n^-(\sigma) \to R_n^-(\sigma) \oplus R_{n+1}^-(\sigma)$, for $n \geq 0$.

Corollary 3.39. *T^- is surjective and T^+ is injective.*

Proof. The first half is directly from the argument of Proposition 3.37. Now we verify that $T^+ : R_n^+(\sigma) \to R_n^+(\sigma) \oplus R_{n+1}^+$ is injective, for $n \geq 1$. It is easy to see $T^-(f_{-n}) = 0$ from the argument of Proposition 3.37 and (12), hence $T^+(f_{-n}) \neq 0$ because we know T is injective from Theorem 4.5.14 in [Abd11]. We are done in this case. The remaining case could be treated in the same way. □

Remark 3.40. *We have indeed determined $T^+(f_{-n})$ in the next Proposition.*

We come to the main result of this subsection:

Proposition 3.41. (1). *If* $\dim \sigma > 1$, *then*

$$Tf_{-m} = f_{-(m+1)},\ m \geq 0;$$
$$Tf_n = f_{n+1},\ n \geq 1.$$

(2). *If* $\dim \sigma = 1$, *say* $\sigma = \eta \circ \det$ *for some character* η *of* k_E^1, *then,*

$$Tf_0 = f_{-1} + \eta(-1) f_1;$$
$$Tf_{-m} = -f_{-m} + f_{-(m+1)},\ m \geq 1;$$
$$Tf_n = -f_n + f_{n+1},\ n > 0.$$

Proof. The proof are tedious calculations, based on Proposition 3.37 and (1) of Proposition 3.7.

We recall the definitions of f_n, for $n \in \mathbb{Z}$,

$$f_{-m} = \sum\nolimits_{i \in N_0/N_{2m}} i\alpha^{-m} \cdot f_0, \text{ for } m \geq 0;$$
$$f_n = \sum\nolimits_{j \in N_1'/N_{2n}'} j\alpha^n\beta \cdot f_0, \text{ for } n > 0.$$

We need to know the initial case, which is already known:

$$Tf_0 = f_{-1} + \lambda_{\beta,\sigma} f_1, \tag{14}$$

say Lemma 3.12, as one can check immediately.

We will show the formula for Tf_{-m} in detail for $m \geq 1$. The Hecke operator respects the I_1-invariants, and from Proposition 3.37, we see there are constants such that

$$Tf_{-m} = c_{-m,-m+1} f_{-m+1} + c_{-m,-m} f_{-m} + c_{-m,-m-1} f_{-m-1}.$$

Let m be an integer bigger than zero. We see at first that:

$$i\alpha^{-m} f_1(\alpha^{m-1}) = \begin{cases} v_0', & i \in N_{2m-2}/N_{2m}, \\ 0, & \text{otherwise.} \end{cases}$$

We also have $f_{-1}(\alpha^{m-1} i \alpha^{-m}) = 0$: when $i \in N_{2m-2}/N_{2m}$ it is clear from the definition of support of f_{-1}; for $i \in N_0 \backslash N_{2m-2}/N_{2m}$, because $d(v_0, \alpha^{m-1} i \alpha^{-m} v_0) > 2$ for $m > 1$, $\alpha^{m-1} i \alpha^{-m}$ is not in the support of f_{-1}. In summary, we have $c_{-m,-m+1}$ is zero.

Next, we calculate $c_{-m,-m}$. Firstly, we have $f_{-1}(\alpha^m i \alpha^{-m}) = 0$, for any $i \in N_0$: for $i \in N_{2m}$, it is clear. For the remaining $i \in N_0 \setminus N_{2m}$, a simple calculation shows that $\alpha^m i \alpha^{-m} \in K_0 \alpha^{-l} I_1$ for some positive l, hence the claim. Secondly, $f_1(\alpha^m i \alpha^{-m})$ vanishes outside of $N_{2m-1} \setminus N_{2m}$. For $i \in N_{2m-1} \setminus N_{2m}$, $f_1(\alpha^m i \alpha^{-m}) = c_i v_0$, where c_i is the value of χ_σ at some specific diagonal element. We write $c = \sum_i c_i$, then

$$c = \begin{cases} -\chi_1'(-1), & \text{if } \chi_\sigma^s = \chi_\sigma = \chi_1 \otimes \chi_2, \\ 0, & \text{otherwise.} \end{cases}$$

In all, we see $c_{-m,-m} = c \cdot \lambda_{\beta,\sigma}$.

Lastly, we come to $c_{-m,-m-1}$. We have in general $f_1(\alpha^{m+1} i \alpha^{-m}) = 0$. Also,

$$f_{-1}(\alpha^{m+1} i \alpha^{-m}) = \begin{cases} v_0, & \text{if } i \in N_{2m}, \\ 0, & \text{otherwise.} \end{cases}$$

We see $c_{-m,-m-1} = 1$.

The calculations for $T \cdot f_n$, $n \geq 1$ work in the same manner, and we get finally that $c_{n,n-1} = 0$, $c_{n,n} = c\lambda_{\beta,\sigma}$, and $c_{n,n+1} = 1$.

As we already know the exact value of $\lambda_{\beta,\sigma}$, we have finished the proof.

□

4 A parametrization theorem

The main concern of this chapter is to prove the following Theorem 4.1. Its content is to match the compactly induced representations and principal series in a precise way. As already mentioned in the Introduction, in the forthcoming work of Abe–Henniart–Herzig–Vignéras, comparison between compact induction and parabolic induction is a major ingredient in their classification of irreducible admissible smooth representations of any p-adic reductive connective group, where their tools are the Satake isomorphism, developed by Herzig and Henniart–Vignéras.

Our approach is again that of Barthel–Livné, i.e., the analysis on the tree is essentially used in our argument, which is the reason that there are some technical difficulties at some places. For simplicity, assume the weight is the trivial representations. After fixing an apartment on the tree, one can associate canonically a unique integer $h(\mathbf{v})$ to any vertex $\mathbf{v}$, which is called the height of the vertex. The Hecke operator T, in the case of GL_2, maps a vertex $\mathbf{v}$ to the sum of vertices which are of distance *one* from the vertex itself. As a result, the unique vertex on the tree, which is of height $h(\mathbf{v})+1$ and adjacent to $\mathbf{v}$, is congruent modulo T to the sum of vertices which are of height $h(\mathbf{v})-1$ and adjacent to $\mathbf{v}$. In the case of $U(2,1)$, the Hecke operator maps a vertex $\mathbf{v}$ of period one to the sum of vertices which are of distance *two* from $\mathbf{v}$, therefore there are several extra vertices appearing in the formula $T\mathbf{v}$ which share the same height as $\mathbf{v}$; as a result, it is not clear in advance one could conclude similarly that the unique vertex, which is of height $h(\mathbf{v})+2$ and of distance *two* from $\mathbf{v}$, is also congruent modulo T to the sum of some vertices which are of height strictly smaller than $h(\mathbf{v})$. A naive way of saving us from this trouble is to use the Hecke operator repeatedly with respect to all the vertices adjacent to and under $\mathbf{v}$. We have carried out the final point in most cases, and in the remaining cases we manage to reduce them to already known cases, hence finally we are done.

We now repeat a little more from the Introduction, as a guide to this chapter.

In 4.2, we reproduce several results on the principal series of G, most of which were proved first in [Abd11], where we mainly follow again the approach of [BL95] and [BL94].

Next, in section 4.3, we prove (1) of Theorem 4.1. As a natural by-product, we have Corollary 4.16. Then in the section 4.4, we show that the compactly induced representation $\mathrm{ind}_{K_0}^G\sigma$ has only irreducible quotients, Proposition 4.17.

We prove the first half of (c) in section 4.5, and in section 4.6 we arrive at a special case of Theorem 4.1, when π has non-trivial K_0-invariant vectors, i.e., π is unramified.

In section 4.7, we modify the strategy of the unramified case to prove a general injective result (Proposition 4.24), which will simplify several arguments in the proofs of (b) and the second half of (c), in the last two sections.

Theorem 4.1. *Assume $\tilde{E}$ is algebraically closed. Let π be an irreducible smooth representation of G and σ be an irreducible sub-representation of $\pi|_{K_0}$. Then,*

(1).(Abdelatif 2011)[7] *The space*

$$\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma,\ \pi)$$

has an eigenvector for the action of the Hecke algebra $\mathscr{H}(K_0, \sigma)$.

(2). *Let λ be an eigenvalue of T in (1). Assume further that:*

$$\lambda \neq \begin{cases} -\chi_1'(-1), & \text{if } \chi_\sigma = \chi_\sigma^s = \chi_1 \otimes \chi_2, \\ 0, & \text{otherwise.} \end{cases}$$

We set a character ε of B such that $\varepsilon|_{H_0} = \chi_\sigma^s$, and

$$\varepsilon(\alpha) = \begin{cases} \lambda + \chi_1'(-1), & \text{if } \chi_\sigma = \chi_\sigma^s, \\ \lambda, & \text{otherwise.} \end{cases}$$

Then, we have the following,

(a). *The space in (1) is one-dimensional.*

(b). *If χ_σ does not factor through determinant, or $\lambda \neq 1 - \chi_1'(-1)$, then we have*

$$\pi \cong \mathrm{ind}_B^G\, \varepsilon.$$

(c). *If χ_σ factors through the determinant, i.e., $\chi_\sigma = \eta \circ \det$ for some character of k_E^1, and $\lambda = 1 - \chi_1'(-1)$, we view η as a character of E^1, by Remark 1.3. Then*

$$\pi \cong \begin{cases} \eta \circ \det, & \text{if } \dim \sigma = 1, \\ \eta \circ \det \otimes Sp, & \text{otherwise.} \end{cases}$$

Here, Sp is the Steinberg representation of G, defined as $\mathrm{ind}_B^G 1/1$.

[7]This is proved in [Abd11], under the assumption that π is admissible.

In view of Theorem 4.1, we give the definition of so-called supersingular representation. Before doing that, we modify T by a constant:

Let T_σ be the following refined Hecke operator: assume χ_σ is the character of I acting on σ^{I_1} and put

$$T_\sigma = \begin{cases} T + \chi_1'(-1), & \text{if } \chi_\sigma = \chi_\sigma^s, \\ T, & \text{otherwise.} \end{cases}$$

Definition 4.2. *An irreducible smooth representation π of G is called supersingular if it is a quotient of $\mathrm{ind}_{K_0}^G \sigma/(T_\sigma)$, for some irreducible smooth representation σ of K_0.*

4.1 Twisting $\mathrm{ind}_{K_0}^G\sigma/(T-\lambda)$ by characters

Let σ be an irreducible smooth representation of K_0, λ be a scalar in $\tilde{E}$. In this section, we record a useful and simple fact, which tells how $\mathrm{ind}_{K_0}^G\sigma/(T-\lambda)$ is changed when twisted by a character of G. Before stating the result, we recall a little more notation as follows.

Write $\chi_\sigma = \chi_{1,\sigma}\otimes\chi_{2,\sigma}$, which is the character of I acting on σ^{I_1}. Let σ_1 be a twist of σ, i.e., $\sigma_1 = \eta\circ\det\otimes\sigma$, for some character η of k_E^1. It is clear $\chi_{\sigma_1} = \chi_\sigma\cdot(\eta\circ\det)$. By Remark 1.3, we may view η as a character of E^1. Therefore, the character $\eta\circ\det$ of K_0 extends to a character $\eta\circ\det$ of G. Also from Remark 1.3, a character of E^1 can be viewed as a character of k_E^1; for a character η' of E^1, the restriction of the character $\eta'\circ\det$ of G to K_0 is just $\eta'\circ\det$. In the definition below, we take another scalar λ_1, with respect to σ and λ.

Definition 4.3. $\lambda_1 = \begin{cases}\lambda+\chi'_{1,\sigma}(-1)-\chi'_{1,\sigma_1}(-1), & \textit{if } \chi_\sigma=\chi_\sigma^s,\\ \lambda, & \textit{otherwise.}\end{cases}$

Lemma 4.4. *We have an isomorphism of G-representations*

$$\mathrm{ind}_{K_0}^G\sigma_1/(T'-\lambda_1)\cong\eta\circ\det\otimes\mathrm{ind}_{K_0}^G\sigma/(T-\lambda),$$

where T and T' are respectively the Hecke operator in $\mathscr{H}(K_0,\sigma)$ and $\mathscr{H}(K_0,\sigma_1)$, defined in Corollary 3.4.

Proof. Any non-zero polynomial $f(T)$ is injective ([Abd11], Théorème 4.5.14) on the compact induction $\mathrm{ind}_{K_0}^G\sigma$. Then the Lemma results from the fact that the following diagram of G-representations is commutative.

$$\begin{tikzcd}
\mathrm{ind}_{K_0}^G\sigma_1 \arrow[r, "\tau"] \arrow[d, "T'-\lambda_1"'] & \eta\circ\det\otimes\mathrm{ind}_{K_0}^G\sigma \arrow[d, "1\otimes(T-\lambda)"] \\
\mathrm{ind}_{K_0}^G\sigma_1 \arrow[r, "\tau"] & \eta\circ\det\otimes\mathrm{ind}_{K_0}^G\sigma
\end{tikzcd}$$

where τ is the isomorphism sending a function $[g,v]$ to the function $[g,\eta\circ\det(g)v]$, for $g\in G, v\in\sigma$. $\square$

4.2 Some results on principal series

Let χ_1 be a character of $E^\times$ and χ_2 be a character of E^1. Let $\mathscr{I}(\chi_1\otimes\chi_2)$ be the underlying space of $\mathrm{ind}_B^G\,\chi_1\otimes\chi_2$, where $\chi_1\otimes\chi_2$ is the character of H, defined by

$$\chi_1\otimes\chi_2:\ H\to\tilde{E}^\times,$$
$$\mathrm{diag}(x,y,\bar{x}^{-1})\mapsto\chi_1(x)\chi_2(y).$$

An application of Lemma 1.2 shows that, for $f \in \mathscr{I}(\chi_1 \otimes \chi_2)$,

$$f(\beta n(x,y)) = \chi_1(\bar{y}^{-1})\chi_2(-\bar{y}y^{-1})f(Id),$$

for y large enough, as f is locally constant.

Let N^* be the subset of $E \times E$, which consists of elements $(x,y) \in E \times E$ such that $y + \bar{y} + x\bar{x} = 0$. Under the operation that $(x,y) \cdot (x_1, y_1) := (x+x_1, y+y_1 - x\bar{x}_1)$, N^* becomes a group, which is naturally isomorphic to N. Denote by N_k^* the image of N_k in N^*, via the former isomorphism, i.e., the subgroup of N^* consisting of elements (x,y) such that $y \in \mathfrak{p}_E^k$, for any integer $k \in \mathbb{Z}$. Let $\mathscr{J}(\chi_1 \otimes \chi_2)$ be the space of locally constant functions φ from N^* to $\tilde{E}$, such that

$$\varphi((x,y)) = c \cdot \chi_1(\bar{y}^{-1})\chi_2(-\bar{y}y^{-1}),$$

for some $c = c(\varphi)$, when y is large enough.

Then we have an isomorphism i of $\tilde{E}$-spaces from $\mathscr{I}(\chi_1 \otimes \chi_2)$ to $\mathscr{J}(\chi_1 \otimes \chi_2)$, which sends f to $i(f)$:

$$i(f)((x,y)) = f(\beta n(x,y)).$$

The inverse j of i is defined as following. For $\varphi \in \mathscr{J}(\chi_1 \otimes \chi_2)$, $j(\varphi) \in \mathscr{I}(\chi_1 \otimes \chi_2)$ is:

$$j(\varphi)(g) = \begin{cases} c(\varphi)\chi_1 \otimes \chi_2(b), \text{ for } g = b \in B, \\ \chi_1 \otimes \chi_2(b)\varphi((x,y)), \text{ for } g = b\beta n(x,y),\ b \in B,\ n(x,y) \in N. \end{cases}$$

For further application, we specify two special functions in $\mathscr{J}(\chi_1 \otimes \chi_2)$, which consist of a basis of the I_1-invariants of $\mathrm{ind}_B^G \chi_1 \otimes \chi_2$. Let g_1 and g_2 be the function in $\mathscr{I}(\chi_1 \otimes \chi_2)$, supported respectively on BI and $B\beta I$, and

$$g_1(Id) = 1,\ g_2(\beta) = 1.$$

Then the images $\varphi_1 = i(g_1)$, $\varphi_2 = i(g_2)$ under the map i are given by;

$$\varphi_1((x,y)) = \begin{cases} 0, & \text{if } \mathrm{val}(y) \geq 0, \\ \chi_1(\bar{y}^{-1})\chi_2(-\bar{y}y^{-1}), & \text{if } \mathrm{val}(y) < 0. \end{cases}$$

$$\varphi_2((x,y)) = \begin{cases} 0, & \text{if } \mathrm{val}(y) < 0, \\ 1, & \text{if } \mathrm{val}(y) \geq 0. \end{cases}$$

In our former notations, $\varphi_2 = 1_{N_0^*}$.

The space $\mathscr{J}(\chi_1 \otimes \chi_2)$ then inherits a G-representation via the above identification.

Lemma 4.5. *Write ε for $\chi_1 \otimes \chi_2$. For $\varphi \in \mathscr{J}(\chi_1 \otimes \chi_2)$,*
(1). $n(x', y')\varphi((x, y)) = \varphi((x + x', y + y' - x\bar{x}'))$
(2). $\alpha\varphi((x, y)) = \varepsilon(\alpha)^{-1}\varphi((\varpi_E x, \varpi_E^2 y))$
(3). $\alpha^{-1}\varphi((x, y)) = \varepsilon(\alpha)\varphi((\varpi_E^{-1} x, \varpi_E^{-2} y))$

Proof. All can be checked easily, by pulling-back to the space $\mathscr{I}(\chi_1 \otimes \chi_2)$. □

Proposition 4.6. *The functions φ_1 and φ_2 generate the whole space $\mathscr{J}(\chi_1 \otimes \chi_2)$.*

Proof. This is from direct calculations.

To be precise, let $\mathscr{S}(N^*, \tilde{E})$ be the space of locally constant functions on N^*, with compact support. By definition, $\mathscr{S}(N^*, \tilde{E})$ is a subspace of $\mathscr{J}(\chi_1 \otimes \chi_2)$, and furthermore, we see

$$\mathscr{J}(\chi_1 \otimes \chi_2) = \mathscr{S}(N^*, \tilde{E}) \oplus \tilde{E}\varphi_1.$$

Also, φ_2 generates the space $\mathscr{S}(N^*, \tilde{E})$, using the lemma above to see a basis of that is obtained by the G-translates of φ_2. We are done. □

Remark 4.7. *In fact, φ_2 is a K_0-translate of φ_1, but the converse is not true. See* (2) *of Proposition 4.14.*

Remark 4.8. *Of course, regarding the result above, one indeed knows that the whole space is generated by any linear combination of φ_1 and φ_2 except the obvious case that $\varepsilon = \chi_1 \otimes \chi_2$ factors through the determinant and the generator is the unique combination of φ_1 and φ_2 (up to a scalar) on which G acts as the character ε, as the principal series is at most length two.*

The following Theorem 4.9 and Proposition 4.10 are already obtained in [Abd11].

Theorem 4.9. *For a character ε of H, which we view as a character of B trivial on N, the principal representation $\mathrm{ind}_B^G \varepsilon$ is irreducible if and only if ε does not factor through the determinant.*

Proof. The 'only if' part is clear. To show the 'if' part, there are mainly two steps. Suppose ε does not factor through determinant. First of all, one shows the I_1-invariants $(\mathrm{ind}_B^G \varepsilon)^{I_1}$ is simple as a natural right $\mathscr{H}_{I_1}$-module. Secondly, one shows the I_1-invariants $(\mathrm{ind}_B^G \varepsilon)^{I_1}$ indeed generate the representation $\mathrm{ind}_B^G \varepsilon$ itself. Then the desired result follows.

The first step is provided by Corollary 2.32. The second step is from Proposition 4.6. □

Proposition 4.10. (1) *Sp is irreducible.*
(2) $Sp^{K_0} = 0$.

Proof. To prove (1), we look at the I_1-invariants of Sp. What we need is to show Sp is generated by the one-dimensional Sp^{I_1}. We analyze firstly the right translation action of I_1 on the coset space $B \setminus G$. To simplify the process, we identify $B \setminus G$ with the set $\{\infty\} \cup N^*$, which we denote by $\bar{N}^*$. Explicitly, B corresponds to ∞, and $B\beta n(x,y)$ corresponds to (x,y), for any (x,y) in N^*. Then $\bar{N}^*$ inherits the right translation action of G on $B \setminus G$. We note furthermore that a function $\varphi \in \mathscr{J}(1\otimes 1)$ can be extended uniquely to a function (which we also denote by φ) in $\mathscr{S}(\bar{N}^*)$, by $\varphi(\infty) = \mathrm{const}(\varphi)$. Here $\mathscr{S}(\bar{N}^*)$ is the space of locally constant functions from $\bar{N}^*$ to $\tilde{E}$ and of compact support. We therefore realize the representation $\mathrm{ind}_B^G(1 \otimes 1)$ on the space $\mathscr{S}(\bar{N}^*) = \mathscr{S}(B \setminus G)$, and also realize the special series Sp on $\mathscr{S}(\bar{N}^*)/\tilde{E}\varphi_0$, where $\varphi_0 = i(g_0)$ for $g_0 = g_1 + g_2$. Denote by 0 the point $(0,0)$ in N^*.

Lemma 4.11. (1). *The right translation action of I_1 on $\bar{N}^*$ has two orbits, O_∞ and O_0, i.e., the orbits of ∞ and the element $(0,0)$, where*

$$O_0 = \{(x,y) \in N^* : \ \mathrm{val}(y) \geq 0\},$$
$$O_\infty = \{\infty,\ (x,y) \in N^* : \ \mathrm{val}(y) < 0\}.$$

(2). *The stabilizer of $(0,0)$ in I_1 is the subgroup of I_1 which consists of lower triangular matrices and it acts transitively on O_∞. The stabilizer of ∞ in I_1 is the subgroup of I_1 which consists of upper triangular matrixes and it acts transitively on O_0.*

Proof. In view of the decomposition $G = BI_1 \cup B\beta I_1$, we see that the orbit of ∞ (resp. $(0,0)$) is the subset O'_∞ (resp. O'_0) of $\bar{N}^*$ which corresponds to the coset space $B \setminus BI_1$ (resp. $B \setminus B\beta I_1$) in the identification mentioned above. Certainly, $B\beta n \in B \setminus B\beta I_1$, for any $n = n(x,y)$, in which $\mathrm{val}(y) \geq 0$. But, from Lemma 1.2, we see $B\beta n \in BI_1$, for any $n = n(x,y)$ in which $\mathrm{val}(y) < 0$. Secondly, we also note another decomposition $G = B \cup B\beta N$, and furthermore $B\beta N = B\beta N_{<0} \cup B\beta N_0$, where $N_{<0} = \{n = n(x,y) : \mathrm{val}(y) < 0\}$. Then (1) is done.

The statements on the stabilizers in (2) are immediate: for $i \in I_1$, $Bi = B$ if and only if i is upper triangular. Also, for an $i \in I_1$, $B\beta i = B\beta$ if and only if i is lower triangular. Note that $BI_1 = BN'_1$, and that $B\beta I_1 = B\beta N_0$. We are done. □

Lemma 4.12. *There is a short exact sequence of I_1-modules :*

$$0 \to \tilde{E}\varphi_0 \to (\mathscr{S}(\bar{N}^*))^{I_1} \to (\mathscr{S}(\bar{N}^*)/\tilde{E}\varphi_0)^{I_1} \to 0,$$

where I_1 acts trivially on φ_0.

Proof. As the I_1-invariant functor is left exact, we only need to show the last map is surjective. Let $\nu \in (\mathscr{S}(\bar{N}^*)/\tilde{E}\varphi_0)^{I_1}$ and let μ be a pull back of ν in $\mathscr{S}(\bar{N}^*)$. We will show μ is also I_1-invariant; in other words, μ is constant on O_∞ and O_0. By definition of pull back, for any $i \in I_1$, there exists a constant $c = c(i) \in \tilde{E}$, such that $i \cdot \mu - \mu = c$. Let (x, y) be any element in O_0. Then from Lemma 4.11, there exist $i \in B \cap I_1$, such that $(x, y) = 0 \cdot i$. By evaluating the former identity at ∞, we see firstly that $c = 0$, which gives us further that $\mu((x, y)) - \mu(0) = i\mu(0) - \mu = 0$, i.e., μ is constant on O_0. Similarly, we can show μ is also constant on O_∞. We have finished the argument. □

We now prove (1) of Proposition 4.10. Let Y be a non-zero G-submodule of $\mathscr{S}(\bar{N}^*)/\tilde{E}\varphi_0$. Denote by Y' the pull back of Y to $\mathscr{S}(\bar{N}^*)$. As I_1 is a pro-p group, $Y^{I_1} \neq 0$. From Lemma 4.12, we also have an exact sequence :

$$0 \to \tilde{E}\varphi_0 \to (Y')^{I_1} \to Y^{I_1} \to 0,$$

from which we have dim $(Y')^{I_1} = 1 + \dim Y^{I_1} \geq 2$. Now from

$$2 = \dim \mathscr{S}(\bar{N}^*)^{I_1} \geq \dim\, (Y')^{I_1} \geq 2,$$

we conclude that $\mathscr{S}(\bar{N}^*)^{I_1} = (Y')^{I_1}$. In particular, $(Y')^{I_1}$ contains 1_{O_0} and 1_{O_∞}. In our former notation, 1_{O_0} and 1_{O_∞} are just respectively the extensions of φ_1 and φ_2, hence Y' contains the subspace of $\mathscr{S}(\bar{N}^*)$ which is extended from the subspace of $\mathscr{J}(1 \otimes 1)$ generated by φ_1 and φ_2, which is nothing but $\mathscr{J}(1 \otimes 1)$ by Proposition 4.6. Therefore, $Y = \mathscr{S}(\bar{N}^*)/\tilde{E}\varphi_0$.

We continue to prove (2) of Proposition 4.10. We note that $\mathscr{I}(1 \otimes 1)^{I_1} = \mathscr{I}(1 \otimes 1)^I$ and $\mathscr{I}(Sp)^I \subset \mathscr{I}(Sp)^{I_1}$, then Lemma 4.12 tells that the following short sequence is exact :

$$0 \to \tilde{E}g_0 \to \mathscr{I}(1 \otimes 1)^I \to \mathscr{I}(Sp)^I \to 0, \tag{15}$$

which gives dim$\mathscr{I}(Sp)^I = 1$. We can conclude that dim$\mathscr{I}(Sp)^{K_0} \leq 1$.

Suppose that $\bar{f} \in \mathscr{I}(Sp)^{K_0}$ and $\bar{f} \neq 0$. Let f be a pull back of $\bar{f}$ in $\mathscr{I}(1 \otimes 1)^I$ (via (15)), so $f \neq 0$. Without loss of generality, we can assume that $f = g_1$. As $\beta \in K_0$, by our assumption, there exists a constant $c \in \tilde{E}$, such that $\beta g_1 = g_1 + c \cdot g_0$. But this is impossible: By evaluating the former equation firstly on the matrix Id, we get that $c \neq 0$. However, when we evaluate the equation at $k = \beta n(0, y)$, where y is an element of $L_1 \setminus \{0\}$, the constant c turns out to be zero, as for the matrix k we have chosen we have

$k \notin BI$ and $k\beta \notin BI$. Contradiction ! We have finished the proof of (2) of Proposition 4.10. $\square$

Remark 4.13. *From the argument above, Sp^{I_1} is generated by the image of g_1.*

Denote by $\bar{g}_1$, $\bar{g}_2$ the image of g_1 and g_2 in the underlying space $\mathrm{ind}_B^G 1/(1)$ of the special series Sp.

Proposition 4.14. (1). *The q^3 dimensional space generated by $\{n'\bar{g}_1,\ n' \in N'_0/N'_1\}$, as a representation of K_0, is isomorphic to the inflation of St.*

(2). *We have the identity*:

$$-\textstyle\sum_{n\in N'_0/N'_1} n\bar{g}_1 = \beta\bar{g}_1.$$

Proof. First, we show the vectors in $\{n'\bar{g}_1,\ n' \in N'_0/N'_1\}$ are indeed linearly independent. But this is immediate: suppose we have constants $l_{n'}$ and a constant c such that

$$\textstyle\sum_{n'\in N'_0/N'_1} l_{n'}n'g_1 = c(g_1+g_2).$$

Comparing the values of the above equation at n' and β respectively, we see all the $l_{n'}$ and c must be 0.

We turn to show the space generated by $\{n'\bar{g}_1,\ n' \in N'_0/N'_1\}$ is K_0-stable. We note that $\bar{g}_1$ is I_1-invariant. As $\{n',\ n' \in N'_0/N'_1 \cup \{\beta\}\}$ consists of a set of representatives for K_0/I, the K_0-representation $\{K_0{\cdot}\bar{g}_1\}$ is linearly generated by $\{n'\bar{g}_1,\ n' \in N'_0/N'_1,\ \beta\bar{g}_1\}$. Hence, we finish the claim, if we could verify the identity in (2):

$$-\textstyle\sum_{n\in N'_0/N'_1} n\bar{g}_1 = \beta\bar{g}_1.$$

Assume there are constants $l_{n'}$, c_β and d, such that,

$$\textstyle\sum_{n'\in N'_0/N'_1} l_{n'}n'g_1 + c_\beta\beta g_1 = d(g_1+g_2)$$

We see the only possibility is that $d = c_\beta = l_{n'}$. In fact, we have the following identity, which holds in general:

$$\textstyle\sum_{n'\in (N'_0\backslash N'_1)/N'_1} n'g_1 + \beta g_1 = g_2.$$

To verify the above equality, we note firstly that both sides of the above equality have the same value at Id and β. Hence, one only needs to verify that the left hand side is also I_1-invariant. This is done by a case by case checking.

In all, we have shown that the space generated by $\{n'\bar{g}_1\}$ is K_0-stable and of dimension q^3. We denote this K_0-representation by R_0. It is easy to see $R_0^{I_1} = \langle \bar{g}_1 \rangle$, also the group I acts trivially on $\bar{g}_1$. Hence R_0 is irreducible and isomorphic to the inflation of St. □

Corollary 4.15. $Soc_{K_0}(Sp) \cong St$

Proof. By the last Proposition, we have a natural K_0-inclusion i from St to Sp, i.e., $i \in \mathrm{Hom}_{K_0}(St, Sp)$. Hence St is contained in $Soc_{K_0}(Sp)$. From Frobenius reciprocity, $\mathrm{Hom}_{K_0}(St, Sp)$ is isomorphic to $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G St,\ Sp)$. The compactly induced representation $\mathrm{ind}_{K_0}^G St$ is generated the I_1-invariant function $[Id, v_0]$. Hence, the space $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G St,\ Sp)$ is one-dimensional, as we have already shown that Sp^{I_1} is one-dimensional in the argument of (2) of Proposition 4.10. Therefore, St appears only once in $Soc_{K_0}(Sp)$.

Let σ be another smooth irreducible representation of K_0, contained in $Soc_{K_0}(Sp)$. Hence it is isomorphic to its K_0-image in Sp. The image is generated by a non-zero vector in Sp^{I_1}, as σ is generated by σ^{I_1}. But (1) of Proposition 4.14 gives that the image is nothing but St. We are done. □

4.3 Proof of (1) of Theorem 4.1

Proof. Before giving the details, we note that the assumption that π *is admissible* in [Abd11] can be removed. The reason here is the key Corollary 3.11 which means we can modify the process of [BL94].

By assumption, we are given a non-zero K_0-embedding ι from σ to $\pi|K_0$. Let ϕ_ι be the corresponding G-morphism in $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma,\ \pi)$ via Frobenius reciprocity.

As $\mathrm{ind}_{K_0}^G\sigma$ is not irreducible, ϕ_ι is not injective, i.e., $\ker\phi_\iota \neq 0$. Hence, $(\ker\phi_\iota)^{I_1} \neq 0$. From the description of Corollary 3.8, there is a character χ (χ_σ or χ_σ^s) such that

$$(\ker\phi_\iota)^{I,\,\chi} \neq 0,$$

in other words, $\mathrm{Hom}_G(\mathrm{ind}_I^G\chi,\ \ker\phi_\iota) \neq 0$. Denote by ϕ_ι^* the map given by the composition with ϕ_ι,

$$\phi_\iota^* :\ \mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \mathrm{ind}_{K_0}^G\sigma) \to \mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \pi).$$

Of course, ϕ_ι^* annihilates $\mathrm{Hom}_G(\mathrm{ind}_I^G\chi,\ \ker\phi_\iota)$, and applying Corollary 3.11 we conclude that the image of ϕ_ι^* in $\mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \pi)$ is a finite dimensional $\mathscr{H}(I,\chi)$-submodule in $\mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \pi)$.

For simplicity, we also denote by ϕ_ι^* the map,

$$\phi_\iota^* :\ \mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_{K_0}^G\sigma) \to \mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \pi).$$

Let $\triangle_0$ be the K_0-morphism in $\mathrm{Hom}_{K_0}(\mathrm{ind}_I^{K_0}\chi, \sigma)$, corresponding to the morphism in $\mathrm{Hom}_I(\chi, \sigma)$ which maps 1 to v_0. We note that $\triangle_0$ is surjective, as σ is irreducible. Inducing these K_0-representations to G, we then get a G-morphism $\triangle$ in $\mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \mathrm{ind}_{K_0}^G\sigma)$ from $\triangle_0$. It is also surjective (for example, see 2.1 of [BL94]).

Then, $\triangle$ induces two composition maps, both denoted by $\triangle^*$:

$$\triangle^*:\ \mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_{K_0}^G\sigma) \to \mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \mathrm{ind}_{K_0}^G\sigma),$$
$$\triangle^*:\ \mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \pi) \to \mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \pi).$$

Therefore, $\triangle^*$ are injective.

It is immediate from the definitions of $\triangle^*$ and ϕ_ι^* that we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \mathrm{ind}_{K_0}^G\sigma) \arrow[r, "\phi_\iota^*"] & \mathrm{Hom}_G\,(\mathrm{ind}_I^G\chi,\ \pi) \\
\mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \mathrm{ind}_{K_0}^G\sigma) \arrow[r, "\phi_\iota^*"] \arrow[u, "\triangle^*"] & \mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \pi) \arrow[u, "\triangle^*"']
\end{tikzcd}$$

From all this, we conclude that $\phi_\iota^*(\mathrm{End}_G(\mathrm{ind}_{K_0}^G\sigma))$ must be a finite dimensional $\mathscr{H}(K_0,\sigma)$-submodule in $\mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \pi)$. As we have shown $\mathscr{H}(K_0,\sigma)$ is a polynomial algebra (Proposition 3.4) , (1) follows. □

We will use the following corollary later:

Corollary 4.16. *Let π be an irreducible smooth representation of G, which is a quotient of some compact induction $\mathrm{ind}_{K_0}^G\sigma$, via the projection θ. Then the $\mathscr{H}(K_0,\sigma)$-submodule $\langle\theta\cdot\mathscr{H}(K_0,\sigma)\rangle$ of $\mathrm{Hom}_G\,(\mathrm{ind}_{K_0}^G\sigma,\ \pi)$ is a finite dimensional $\tilde{E}$-space.*

Proof. Directly from the argument of the last result. □

It will take a while to prove (2) of Theorem 4.1 completely. In the next section, we insert an interesting by-product of the last corollary.

4.4 $\mathrm{ind}_{K_0}^G\sigma$ has only irreducible quotients

In this section, we record a simple observation on the subrepresentations of $\mathrm{ind}_{K_0}^G\sigma$. The main result is the following proposition, whose analogue for $GL(2)$ seems to be well-known to experts, though we didn't find it in literature.

Proposition 4.17. *Any non-zero subrepresentation of $\mathrm{ind}_{K_0}^G\sigma$ is non-admissible and reducible, of infinite length.*

Proof. The first half is indeed a corollary of Proposition 3.11. Let π be a non-zero subrepresentation of $\mathrm{ind}_{K_0}^G\sigma$. Then π^{I_1} is a non-zero submodule of $(\mathrm{ind}_{K_0}^G\sigma)^{I_1}$ over $\mathscr{H}_{I_1}$. Hence, we are given a non-zero submodule $\pi^{I,\chi}$ of $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$ over $\mathscr{H}(I,\chi)$, for some character χ of I (One certainly can say what the character χ is). Now Proposition 3.11 tells that $\pi^{I,\chi}$ is of finite codimension in $(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$, and it follows that $\dim \pi^{I,\chi} = \infty$ from the infinitude of $\dim(\mathrm{ind}_{K_0}^G\sigma)^{I,\chi}$.

Now we prove the second half of the Proposition. Assume π is an irreducible subrepresentation of $\mathrm{ind}_{K_0}^G\sigma$. Denote the inclusion by ι. As a smooth representation of G, π contains an irreducible smooth representation σ' of K_0. Frobenius reciprocity gives a non-zero G-morphism θ from $\mathrm{ind}_{K_0}^G\sigma'$ to π. In particular, the composition $\iota\circ\theta$ is non-zero. Now θ will factorize through a non-constant polynomial $P(T')$, i.e., $\theta\circ P(T') = 0$, where $P(T')\in\mathscr{H}(K_0,\sigma')$, by Corollary 4.16 (It is here that we really use the assumption that π is irreducible). We have now $(\iota\circ\theta)\circ P(T') = \iota\circ(\theta\circ P(T')) = 0$. But this is impossible because $P(T')$ and $\iota\circ\theta$ are both non-zero, and we have $(\iota\circ\theta)\circ P(T') = P'(T)\circ(\iota\circ\theta)$ for another non-zero polynomial P', from (2) of Corollary 3.5. Now a contradiction arises from the injectivity of T (hence of any non-zero $f(T)$) ([Abd11], Théorème 4.5.14)[8]. $\square$

4.5 The subquotients of V_0

Assume σ is the trivial representation of K_0. Hence, we identify the underlying space $\mathscr{I}(K_0)$ of the compact induction $\mathrm{ind}_{K_0}^G 1$ with the space $C_0(\triangle^1)$ of 0-chains of period one, i.e., the space of all finite linear combination $\sum t_v v$ for period one vertices. Let Deg be the map from $C_0(\triangle^1)$ to $\tilde{E}$: $\mathrm{Deg}(c) = \sum\alpha_v$, for $c = \sum_v \alpha_v\cdot v$, where $\alpha_v\in\tilde{E}$. This map is a surjective G-morphism and trivial on $T(\mathscr{I}(K_0))$. We denote by $\overline{\mathrm{Deg}}$ the induced map. The following proposition is already stated without proof in [Abd11].

Proposition 4.18. (1). *The kernel of $\overline{\mathrm{Deg}}$ is isomorphic to the special series Sp.*

(2). *The induced short exact sequence is non-split :*

$$0\to Sp\to V_0\to Triv\to 0,$$

where we write $\mathscr{I}(K_0)/(T)$ as V_0.

Proof. We prove (2) at first. The method here follows [BL95].

[8] One can indeed conclude a contradiction from the argument of Corollary 3.5: the bimodule structures described there guarantee that there is no non-trivial annihilator.

Suppose the sequence is split. Then by definition, we have a pull-back $\bar{c}$ of $1 \in \tilde{E}$, which is G-invariant. Let $c \in \mathscr{I}(K_0)$ be a representative of $\bar{c}$. Hence, $g \cdot c - c \in T(\mathscr{I}(K_0))$ for any $g \in G$. Assume the support of c is contained in the ball $B_{2k}(\mathbf{v}_0)$ $(= \cup_{0 \le l \le k} C_l)$ for some integer $k \geq 0$. Take $g = \alpha^{2k+1}$. For a 0-chain $a \in C_0(\triangle^1)$, let $\overline{\text{supp}\, a}$ be the set of period one vertices of the minimal subtree of $\triangle$ containing $\text{supp}\, a$. We see $\overline{\text{supp}\,(g \cdot c - c)} \subset B_{2k}(\mathbf{v}_0) \cup \alpha^{2k+1} B_{2k}(\mathbf{v}_0) = B_{2k}(\mathbf{v}_0) \cup\ B_{2k}(\mathbf{v}_{4k+2})$, which we denote by X. Write $g \cdot c - c = T\, b$ for some 0-chain $b \in C_0(\triangle^1)$. We then claim that $\text{supp}\, b \subset X - \{\mathbf{v}_{2k},\ \mathbf{v}_{2k+2}\}$. We observe firstly that it is contained in X, from the definition of T and that of the minimal subtree. Secondly, for $\mathbf{v} = \mathbf{v}_{2k}$, or $\mathbf{v}_{2k+2}$, there is some vertex $\mathbf{v}'$ which is distance 2 from $\mathbf{v}$, and is not in X. *However*, we can always choose such a $\mathbf{v}'$ which is not a neighbour of $\mathbf{v}_{2k+1}$. Then If $\mathbf{v}$ is in $\text{supp}\, b$, $\mathbf{v}'$ would definitely lie in $\text{supp}\,(g \cdot c - c)$, a contradiction. Therefore, it is safe to write b as a unique sum $b_1 + b_2$ of two 0-chains, where $\text{supp}\, b_1 \subset X_1 = B_{2k}(\mathbf{v}_0) - \mathbf{v}_{2k}$ and $\text{supp}\, b_2 \subset X_2 = B_{2k}(\mathbf{v}_{4k+2}) - \mathbf{v}_{2k+2}$. As now $d(X_1,\ X_2) \geq 6$, $\text{supp}\,(T\, b_1)$ and $\text{supp}\,(T\, b_2)$ are disjoint. Hence, by comparing the supports, $T\, b_1 = -c$, i.e., $\bar{c} = 0$.

To prove (1), we need some preparation, which also paves the way to the proof of the unramified case of Theorem 4.1.

Let Λ be a variable, and set $R = \tilde{E}[\Lambda, \Lambda^{-1}]$. Define an unramified character $X : E^\times \to R^\times$, by $X(\varpi_E) = \Lambda^{-1}$. We form the character $X \otimes 1$ of T by: $X \otimes 1(t) = X(x)$, where t is the matrix:

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \bar{x}^{-1} \end{pmatrix}.$$

Then we view $X \otimes 1$ as a character of B which is trivial on the subgroup N. The character we choose above guarantees the existence of a non-trivial function f_0 in the former space which is K_0-invariant, i.e., by writing an element g as bk, $f_0(bk) = X \otimes 1(b)$, where $b \in B,\ k \in K_0$. Now we transfer the result of first subsection to our situation. Define $\mathscr{J}(X \otimes 1)$ as the space of locally constant functions φ from N^* to R which satisfy $\varphi((x, y)) = c \cdot \Lambda^{\text{val}(y)}$ for some constant $c = \text{const}(\varphi) \in R$, when y is large enough. Therefore by the above, the map i which maps a f to $i(f)$, where $i(f)((x, y)) = f(\beta n(x, y))$, is an isomorphism from $\mathscr{I}(X \otimes 1)$ to $\mathscr{J}(X \otimes 1)$. The inverse j of i explicitly: for a function φ in $\mathscr{J}(X \otimes 1)$

$$j(\varphi)(g) = \begin{cases} \text{const}(\varphi) X \otimes 1(b), & \text{when } g = b \in B, \\ X \otimes 1(b) \varphi((x, y)), & \text{when } g = b\beta n(x, y). \end{cases} \tag{16}$$

The space $\mathscr{J}(X\otimes 1)$ then inherits a structure of G-module. We record the function φ_0:

$$\varphi_0((x,y)) = \begin{cases} \Lambda^{\mathrm{val}(y)}, & \text{if } \mathrm{val}(y)\le 0,\\ 1, & \text{if } \mathrm{val}(y)\ge 0.\end{cases}$$

Let $\mathscr{S}(N^*,R)$ be the space of locally constant functions on N^*, which take values in R and have compact support. By definition, $\mathscr{S}(N^*,R)$ is a subspace of $\mathscr{J}(X\otimes 1)$, which has a set $\{1_{N_k^*\cdot(x,y);\ k\in\mathbb{Z},\ (x,y)\in N^*}\}$ of characteristic functions as generators, and there exists a direct sum decomposition: $\mathscr{J}(X\otimes 1)=\mathscr{S}(N^*,R)\oplus R\varphi_0$.

Lemma 4.19. *For $\varphi\in\mathscr{J}(X\otimes 1)$,*

(1). $n(x',y')\varphi((x,y))=\varphi((x+x',y+y'-x\bar{x}'))$.
(2). $\alpha\varphi((x,y))=\Lambda^{-1}\varphi((\varpi_E x,\varpi_E^2 y))$.
(3). $\alpha^{-1}\varphi((x,y))=\Lambda\varphi((\varpi_E^{-1}x,\varpi_E^{-2}y))$.

Proposition 4.20. $\varphi_0\mid T=(\Lambda-1)\varphi_0$

Proof. This is in fact a simpler variant of Proposition 3.24. □

As f_0 is K_0-invariant, φ_0 is also K_0-invariant. This K_0-invariant vector gives rise to a G-morphism $\phi_{\varphi_0}^{K_0}$ from $\mathrm{ind}_{K_0}^G 1$ to $\mathrm{ind}_B^G X\otimes 1$ which corresponds to φ_0 by Frobenius reciprocity, i.e., $\phi_{\varphi_0}^{K_0}(1_{K_0})=\varphi_0$. This morphism extends to an R-linear morphism from the representation space $\mathscr{V}$ of $\mathrm{ind}_{K_0}^G 1\otimes_{\tilde{E}} R$ to $\mathscr{J}(X\otimes 1)$, which we also denote by $\phi_{\varphi_0}^{K_0}$.

We are interested in the properties of $\phi_{\varphi_0}^{K_0}$. Firstly, we determine its image in $\mathscr{J}(X\otimes 1)$. Before doing this, we compute a special G-translation of φ_0.

Proposition 4.21. $\sum_{y\in L_1} n(0,\varpi_E^{-1}y)\varphi_0=(1-\Lambda^{-1})1_{N_{-1}^*}$

Proof. For $(x_1,y_1)\in N^*$,
$$\begin{aligned}\sum_{y\in L_1} n(0,\varpi_E^{-1}y)\varphi_0((x_1,y_1)) &\\ =\sum_{y\in L_1} f_0(\beta n(x_1,y_1+\varpi_E^{-1}y)).&\end{aligned}$$

Now we apply Lemma 1.2. When $\mathrm{val}(y_1)\ge 0$, $n(x_1,y_1)\in K_0$. In this case, the term $y=0$ gives value 1, and for other terms Lemma 1.2 gives that each $f_0(\beta n(0,\varpi_E^{-1}y))$ is Λ^{-1}. So the sum is $1+(q-1)\Lambda^{-1}=1-\Lambda^{-1}$.

When $\mathrm{val}(y_1)\le -2$, Lemma 1.2 again gives that each $f_0(\beta n(x_1,y_1+\varpi_E^{-1}y))$ is $\Lambda^{\mathrm{val}(y_1)}$. So in this case, the sum is zero.

For the remaining case of $\mathrm{val}(y_1)=-1$, there exists a unique $y\in L_1$ so that $y+\varpi_E y_1$ is in $\mathfrak{p}_E$. This term gives value 1. For other terms, Lemma

1.2 also gives that each $f_0(\beta n(x_1, y_1 + \varpi_E^{-1}y))$ is Λ^{-1}. The sum turns out to be $1-\Lambda^{-1}$. We are done. $\square$

Theorem 4.22. *(1). The image of $\mathscr{V}$ under $\phi_{\varphi_0}^{K_0}$ is $(1-\Lambda^{-1})\mathscr{S}(N^*, R) \oplus R\varphi_0$.*

(2). The kernel of $\phi_{\varphi_0}^{K_0}$ is $(T-\Lambda+1)\mathscr{V}$.

Proof. We deal with (1) first. From (3) of Lemma 4.19, we get $\alpha^{-n}1_{N^*_{-1}} = \Lambda^n 1_{N^*_{2n-1}}$. Then for any integer n, $(1-\Lambda^{-1})1_{N^*_{2n-1}}$ is in the image of $\phi_{\varphi_0}^{K_0}$ by Proposition 4.21.

By (1) of Lemma 4.19, $n(x,y)1_{N^*_l} = 1_{N^*_l(x,y)^{-1}}$. This shows that, for any $(x,y) \in N^*$ and any integer n, $(1-\Lambda^{-1})1_{N^*_{2n-1}\cdot(x,y)}$ lies in the image of $\phi_{\varphi_0}^{K_0}$. Furthermore, we have

$$1_{N^*_{2n}} = \sum_{(x,y)\in L_2} n(\varpi_E^n x, \varpi_E^{2n} y)1_{N^*_{2n+1}}, \tag{17}$$

and using (1) of Lemma 4.19 again, we see that, for any $(x,y) \in N^*$ and any integer n, $(1-\Lambda^{-1})1_{N^*_{2n}\cdot(x,y)}$ lies in the image of $\phi_{\varphi_0}^{K_0}$. We have proved $(1-\Lambda^{-1})\mathscr{S}(N^*, R) \oplus R\varphi_0$ is contained in the image of $\phi_{\varphi_0}^{K_0}$.

Now for a vertex $\mathbf{v} \in \triangle^1$, there is a unique path from $\mathbf{v}$ to $\mathbf{v}_0$; as a result we could express $\mathbf{v}$ as $\sum_l t_l(\mathbf{v}_l - \Lambda^{-1}a^2(\mathbf{v}_l)) + t_0\mathbf{v}_0$, where $t \in R$ and $a^2(\mathbf{v}_l)$ is the unique vertex which is of distance 2 from $\mathbf{v}_l$ and with height $h(\mathbf{v}_l)+2$. This expression of $\mathbf{v}$ changes into $\sum_l t_l g_l(\mathbf{v}_0 - \Lambda^{-1}\mathbf{v}_2) + t_0\mathbf{v}_0$ for some g_l in G (we have used Remark 3.30 here). Then $\phi_{\varphi_0}^{K_0}(\mathbf{v}) = \sum_l t_l g_l(\varphi_0 - \Lambda^{-1}\alpha\varphi_0) + t_0\varphi_0$.

By the definition of φ_0 and (2) of Lemma 4.19. we compute $\varphi_0 - \Lambda^{-1}\alpha\varphi_0 = (1-\Lambda^{-1})(\Lambda^{-1}1_{N^*_{-1}} + 1_{N^*_0})$. We also note that $g\varphi$ is in $\mathscr{S}(N^*, R) \oplus R\varphi_0$, for any $g \in G$ and $\varphi \in \mathscr{S}(N^*, R)$. This shows that $\phi_{\varphi_0}^{K_0}(\mathbf{v})$ is in the space $(1-\Lambda^{-1})\mathscr{S}(N^*, R) \oplus R\varphi_0$. This finishes our argument.

We now prove (2). Firstly, by Proposition 4.20, we have $\phi_{\varphi_0}^{K_0}((T-\Lambda+1)(1_{K_0})) = 0$. As the G-translates of 1_{K_0} generate $\mathscr{V}$, we conclude that $\phi_{\varphi_0}^{K_0}$ vanishes on $(T-\Lambda+1)\mathscr{V}$.

Given $c \in \mathscr{V}$ such that $\phi_{\varphi_0}^{K_0}(c) = 0$, we write c as $\sum_{\mathbf{v}\in S} t_{\mathbf{v}} \cdot \mathbf{v}$, where S is a finite set of $\triangle^1$. So we can find a vertex $\mathbf{v}_{2r}$ in the standard apartment such that $\mathbf{v}_{2r} \in \cap_{\mathbf{v}\in S} \overline{\mathbf{v}\infty}$, i.e., all the vertices in S are under $\mathbf{v}_{2r}$. Put $2s = \min_{\mathbf{v}\in S} h(\mathbf{v})$. Then if we allow some $t_{\mathbf{v}}$ to be zero, we can assume S to be the finite subset of $\triangle^1$ consisting of all the vertices under $\mathbf{v}_{2r}$ and with height greater than or equal to $2s$.

Step 1 There is an equality:

$$\mathbf{v} = \Lambda^{-1}a^2(\mathbf{v}) - \Lambda^{-1}(a^2(\mathbf{v}) - \Lambda\mathbf{v}).$$

Replacing each $\mathbf{v} \in S$ in the expression of c by the right side of the equality above (doing this from the vertices of least height and moving up), we get:

$$c = P \cdot \mathbf{v}_{2r} + \sum_{\mathbf{v} \in S, \mathbf{v} \neq \mathbf{v}_{2r}} P_{\mathbf{v}} \cdot (a^2(\mathbf{v}) - \Lambda \mathbf{v}), \tag{18}$$

where P and $P_{\mathbf{v}}$ are polynomials in Λ, Λ^{-1}.

Step 2 Recall that for $\mathbf{v} \in \triangle^1$, $T\mathbf{v} = \sum_{d(\mathbf{v}',\mathbf{v})=2} \mathbf{v}'$. Then we get that for such a $\mathbf{v}$,

$$T\mathbf{v} + \mathbf{v} = a^2(\mathbf{v}) + \sum_{a(\mathbf{v}')=a(\mathbf{v})} \mathbf{v}' + \sum_{a^2(\mathbf{v}')=\mathbf{v}} \mathbf{v}'.$$

Note that the numbers of terms appearing in the second and the third sum above are respectively q and q^4. Then a rearrangement gives

$$T\mathbf{v}+\mathbf{v}-\Lambda\mathbf{v} = a^2(\mathbf{v})-\Lambda\mathbf{v}+\Lambda^{-1} \sum_{a(\mathbf{v}')=a(\mathbf{v})} (\Lambda\mathbf{v}'-a^2(\mathbf{v}'))+\Lambda^{-1} \sum_{a^2(\mathbf{v}')=\mathbf{v}} (\Lambda\mathbf{v}'-a^2(\mathbf{v}')). \tag{19}$$

Equivalently ,

$$a^2(\mathbf{v}) - \Lambda\mathbf{v} \equiv \Lambda^{-1} \sum_{a(\mathbf{v}')=a(\mathbf{v})} (a^2(\mathbf{v}') - \Lambda\mathbf{v}') + \Lambda^{-1} \sum_{a^2(\mathbf{v}')=\mathbf{v}} (a^2(\mathbf{v}') - \Lambda\mathbf{v}'). \tag{20}$$

where the congruences appearing above and below are all $\bmod (T - \Lambda + 1)\mathscr{V}$.

Note that in the first sum on the right of (20), $\mathbf{v}'$ goes through all the vertices under and adjacent to $a(\mathbf{v})$, which particularly means that these $\mathbf{v}'$ are of the same height. So the height is not reduced if we insert (20) directly into the expression (18) of c that we got in step one.

Now write $a(\mathbf{v})$ as $\mathbf{u}$. Viewing $\mathbf{u}$ as fixed, we sum (20) over the vertices $\mathbf{v}''$ which are under and adjacent to $\mathbf{u}$. Then the first sum on the right of (20) disappears as it becomes a constant and is counted q times. We get

$$\begin{aligned} \sum_{a(\mathbf{v}'')=\mathbf{u}} (a^2(\mathbf{v}'') - \Lambda\mathbf{v}'') &\equiv \sum_{a(\mathbf{v}'')=\mathbf{u}} \Lambda^{-1} \sum_{a^2(\mathbf{v}')=\mathbf{v}''} (a^2(\mathbf{v}') - \Lambda\mathbf{v}') \\ &\equiv \Lambda^{-1} \sum_{a^3(\mathbf{v}')=\mathbf{u}} (a^2(\mathbf{v}') - \Lambda\mathbf{v}') \end{aligned}$$

Then by inserting the above into the right hand of (20), we finally obtain

$$a^2(\mathbf{v}) - \Lambda\mathbf{v} \equiv \Lambda^{-2} \sum_{a^3(\mathbf{v}')=a(\mathbf{v})} (a^2(\mathbf{v}') - \Lambda\mathbf{v}') + \Lambda^{-1} \sum_{a^2(\mathbf{v}')=\mathbf{v}} (a^2(\mathbf{v}') - \Lambda\mathbf{v}'). \tag{21}$$

We also note that the right hand of (21) can be written as

$$\sum_{a^3(\mathbf{v}')=a(\mathbf{v})} Q_{\mathbf{v}'} \cdot (a^2(\mathbf{v}') - \Lambda\mathbf{v}'),$$

where $Q_{\mathbf{v}'}$ is some polynomial in Λ and Λ^{-1} (depending on $\mathbf{v}'$).

Using (21) for all $\mathbf{v} \in S$ except $\mathbf{v}_{2r}$ (starting from the top and going down, in (18)), we get

$$c \equiv P \cdot \mathbf{v}_{2r} + \sum_{\mathbf{v}\in S,\ h(\mathbf{v})=2s} P'_{\mathbf{v}} \cdot (a^2(\mathbf{v}) - \Lambda\mathbf{v}).$$

Our assumption is $\phi^{K_0}_{\varphi_0}(c) = 0$. Then by Proposition 4.20, the congruence above gives

$$0 = P \cdot \phi^{K_0}_{\varphi_0}(\mathbf{v}_{2r}) + \sum_{\mathbf{v}\in S,\ h(\mathbf{v})=2s} P'_{\mathbf{v}} \cdot \phi^{K_0}_{\varphi_0}(a^2(\mathbf{v}) - \Lambda\mathbf{v}). \tag{22}$$

We need to compute the right hand side of the equation more explicitly. Firstly,

$$\phi^{K_0}_{\varphi_0}(\mathbf{v}_{2r}) = \phi^{K_0}_{\varphi_0}(\alpha^r\mathbf{v}_0) = \alpha^r\varphi_0.$$

Secondly, from the proof of Lemma 3.26, we know that

$$\{v \in S | h(\mathbf{v}) = 2s\} = (N_{-2r}/N_{-2s})\mathbf{v}_{2s}.$$

Then given $n = n(x, y) \in N_{-2r}$, which means that $y \in \mathfrak{p}_E^{-2r}$, from Lemma 3.29 and Lemma 4.19 we see that

$$\begin{aligned}
\phi^{K_0}_{\varphi_0}(a^2(n\mathbf{v}_{2s}) - \Lambda n\mathbf{v}_{2s}) &= \phi^{K_0}_{\varphi_0}(n\mathbf{v}_{2s+2} - \Lambda n\mathbf{v}_{2s}) \\
&= \phi^{K_0}_{\varphi_0}(n\alpha^s(\alpha\mathbf{v}_0 - \Lambda\mathbf{v}_0)) \\
&= n\alpha^s(\alpha\varphi_0 - \Lambda\varphi_0) \\
&= (\Lambda^{-1} - 1)\Lambda^{-s}(1_{N^*_{-2s-1}\cdot(x,y)^{-1}} + \Lambda \cdot 1_{N^*_{-2s}\cdot(x,y)^{-1}}),
\end{aligned}$$

from which it turns out that the supports of the functions $\phi^{K_0}_{\varphi_0}(a^2(n\mathbf{v}_{2s}) - \Lambda n\mathbf{v}_{2s})$ may intersect as $n = n(x, y)$ goes through N_{-2r}/N_{-2s}.

One observes from these computations that $\phi^{K_0}_{\varphi_0}(\mathbf{v}_{2r})$ is of non-compact support, but all the other $\phi^{K_0}_{\varphi_0}(a^2(\mathbf{v})-\Lambda\mathbf{v})$ have compact support. Therefore we can conclude that $P \equiv 0$. Then by substituting the display above, (22) turns into

$$\sum_{n=n(x,y)\in N_{-2r}/N_{-2s}} P'_{n(x,y)} \cdot (1_{N^*_{-2s-1}\cdot(-x,\bar{y})} + \Lambda \cdot 1_{N^*_{-2s}\cdot(-x,\bar{y})}) = 0,$$

where we write $P'_{n(x,y)}$ for $P'_{\mathbf{v}}$, for $\mathbf{v} = n(x, y)\mathbf{v}_{2s}$. Note that $(0, \varpi_E^{-2s-1}a)$ commutes with (x, y)(as elements in N^*). When decomposing N^*_{-2s-1} into

$\cup_{l\in L_1} N^*_{-2s}(0,\varpi_E^{-2s-1}l)$, and re-writing the sum over the left cosets, the above equation turns into

$$\sum_{\substack{n\in N_{-2s}\backslash N_{-2r},\\ n=n(x,y)}} P'_{n(-x,\bar{y})}\cdot\left((1+\Lambda)1_{N^*_{-2s}\cdot(x,y)}+\sum_{i=1}^{q-1}1_{N^*_{-2s}\cdot(x,y+\varpi_E^{-2s-1}l_i)}\right)=0, \tag{23}$$

in which $\{l_i, 1\le i\le q-1\}=L_1\setminus\{0\}$.

For simplicity, we will rewrite $P'_{n(-x,\bar{y})}$ above as $P''_{n(x,y)}$. To deal with (23), we note first that $n(x',y'+\varpi_E^{-2s-1}l)$ goes through $N_{-2s}\backslash N_{-2r}$ when $n(x',y')$ and l go through $N_{-2s-1}\backslash N_{-2r}$ and L_1 respectively.

Then another observation we need is that: for a given $n=n(x,y)\in N_{-2s}\backslash N_{-2r}$, $N^*_{-2s-1}\cdot(x,y)$ is fixed by $(0,\varpi_E^{-2s-1}l)$ for $l\in L_1$, and moreover when l goes through L_1, $N^*_{-2s}\cdot(x,y+\varpi_E^{-2s-1}l)$ also goes through $N^*_{-2s}\cdot(x,y)\backslash N^*_{-2s-1}\cdot(x,y)$.

With these in mind, we can see that for a fixed $(x',y')\in N^*_{-2s-1}\backslash N^*_{-2r}$, the coefficient of a characteristic function $1_{N^*_{-2s}\cdot(x',y'+\varpi_E^{-2s-1}l_i)}$ (appearing in (23)) is $(1+\Lambda)P''_i+\sum_{j\neq i}P''_j$, where P''_i (relative to (x',y')) is short for $P''_{n(x',y'+\varpi_E^{-2s-1}l_i)}$. Therefore we can rewrite (23) as:

$$\sum_{\substack{n\in N_{-2s-1}\backslash N_{-2r},\\ n=n(x',y')}}\sum_{i=0}^{q-1}((1+\Lambda)P''_i+\sum_{j\neq i}P''_j)\cdot 1_{N^*_{-2s}\cdot(x',y'+\varpi_E^{-2s-1}l_i)}=0. \tag{24}$$

Now from (24) we arrive to conclude that for a fixed $(x',y')\in N^*_{-2s-1}\backslash N^*_{-2r}$,

$$(1+\Lambda)P''_i+\sum_{j\neq i}P''_j=0,\ \ 0\le i\le q-1. \tag{25}$$

Then it is a matter to solve for P''_i from (25). In fact, by adding together all the equations in (25), we get $\sum_{i=0}^{q-1}(1+\Lambda+q-1)P''_i=0$, which is just

$$\sum_{i=0}^{q-1}P''_i=0. \tag{26}$$

Subtracting (26) from every equation in (25), we obtain that all the P''_i are 0.

Changing back the notations, we have indeed shown that $P''_{n(x,y)}$ are all 0, for $n(x,y)\in N_{-2s}\backslash N_{-2r}$, i.e., $P'_{n(x,y)}$ are all 0, for $n(x,y)\in N_{-2r}/N_{-2s}$. We have finally proved $c\equiv 0$, i.e., $c\in(T-\Lambda+1)\mathscr{V}$. We are done. □

Now we are ready to prove (1) of Proposition 4.18.

Let $\mathscr{V}'$ be the underlying space of the representation $\mathrm{ind}_{K_0}^G 1 \otimes_{\mathcal{H}_{K_0}} R$. Then we have an isomorphism

$$\mathscr{V}'/T\mathscr{V}' \cong V_0 = \mathscr{I}(K_0)/T\mathscr{I}(K_0).$$

Hence, we are given a degree map:

$$\overline{\mathrm{Deg}}: \ V_0 = \mathscr{V}'/T\mathscr{V}' \to \tilde{E}. \tag{27}$$

We now apply the *Lemma 31* of [BL95] to our situation: $D = R$, $P = (\Lambda - 1)$, $S =$ the group algebra $\tilde{E}[G]$, $Y' = \mathscr{V}'$, $Y = \mathscr{J}(X \otimes 1)$, hence we view both Y and Y' as a $(S,\ R)$-bi-module. Then, we use Theorem 4.22:

We indeed have $\mathscr{V}'/T\mathscr{V}' = \mathscr{V}'/(\Lambda - 1)\mathscr{V}'$, from Proposition 4.20 . On the other hand, $(\Lambda - 1)\mathscr{J}(X \otimes 1)$ is contained in the image of $\mathscr{V}'$ under the injection $\phi_{\varphi_0}^{K_0}$ by (1) of Theorem 4.22. So the condition of *Lemma 31* of [BL95] is satisfied. As an $\tilde{E}[G]$-module, $\mathscr{V}'/T\mathscr{V}'$ and $\mathscr{J}(X \otimes 1)/(\Lambda - 1)\mathscr{J}(X \otimes 1)$ have the same length and the same Jordan-Hölder factors. However, $\mathscr{J}(X \otimes 1)/(\Lambda - 1)\mathscr{J}(X \otimes 1)$ is just $\mathscr{J}(1 \otimes 1)$, i.e., the space of the representation $\mathrm{ind}_B^G 1 \otimes 1$, which is of length 2 from Proposition 4.10, with two Jordan-Hölder factors: Triv, Sp. Hence, the Kernel of $\overline{\mathrm{Deg}}$, as an $\tilde{E}[G]$-module, must be irreducible and isomorphic to the special series Sp. □

4.6 Unramified case

For a non-zero $\lambda \in \tilde{E}$, let χ_λ be the unramified character of $E^\times$, which takes value λ^{-1} at ϖ_E. The preparation in last subsection leads to a special case of Theorem 4.1.

Theorem 4.23. *Let (π, V) be an irreducible smooth representation of G such that $V^{K_0} \neq 0$. Then,*

(1). *There exist a vector $v \neq 0$ in V^{K_0} which is an eigenvector for $\mathscr{H}_{K_0}$.*

(2). *Let v be an eigenvector in* (1)*, and denote by λ the corresponding eigenvalue, i.e., $v \mid T = \lambda v$. Suppose $\lambda \neq -1$. Then,*

(*a*). *If $\lambda \neq 0$, then* $\mathrm{dim} V^{K_0} = 1$ *and* $(\pi, V) \cong \mathrm{ind}_B^G \chi_{\lambda+1} \otimes 1$;

(*b*). *If $\lambda = 0$, then* $\mathrm{dim} V = 1$, *and* $(\pi, V) \cong \mathrm{Triv}$.

Proof. (1). We have indeed proved this in general.

(2). For v as in (1), denote by λ the corresponding eigenvalue, i.e., $v \mid T = \lambda v$. Assume that $\lambda \neq -1$. By the definition of the right action, $\phi_v^{K_0}$ is trivial on $(T - \lambda)\mathscr{I}(K_0)$. So (π, V) is equivalent to an irreducible quotient of

$$\mathrm{ind}_{K_0}^G 1/(T - \lambda)\mathrm{ind}_{K_0}^G 1$$

via the map $\phi_v^{K_0}$.

For (b), where $\lambda = 0$. By Proposition 4.18, $\mathrm{ind}_{K_0}^G 1/(T)\mathrm{ind}_{K_0}^G 1$ contains the special series Sp, with quotient Triv. As Sp is the unique subrepresentation of $\mathrm{ind}_{K_0}^G 1/(T)\mathrm{ind}_{K_0}^G 1$ (from (2) of 4.18), we conclude that $\pi \cong$ Triv.

For (a), where $\lambda \neq 0$. As $\lambda + 1 \neq 0$, we can form the principal series $\mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1)$ with underlying space $\mathscr{J}(\chi_{\lambda+1} \otimes 1)$, and it is irreducible as $\lambda + 1 \neq 1$. The K_0-invariant function φ_0 in $\mathscr{J}(\chi_{\lambda+1} \otimes 1)$ gives rise to a G-morphism $\phi_{\varphi_0}^{K_0}$ from $\mathrm{ind}_{K_0}^G 1$ to $\mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1)$. From Proposition 4.20, we see $\varphi_0 \mid T = \lambda\varphi_0$. Hence $\phi_{\varphi_0}^{K_0}$ is trivial on $(T-\lambda)\mathrm{ind}_{K_0}^G 1$ and we get an induced morphism:

$$\phi_{\varphi_0}^{K_0} : \mathrm{ind}_{K_0}^G 1/(T-\lambda)\mathrm{ind}_{K_0}^G 1 \to \mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1). \tag{28}$$

Now the right side of the above is irreducible. From the conditions that $\lambda + 1 \neq 1$ and $\lambda + 1 \neq 0$, the same argument (changing Λ into $\lambda + 1$) in proving (2) of Theorem 4.22 will imply that the $\phi_{\varphi_0}^{K_0}$ above is injective. But it is surely non-zero. Therefore $\phi_{\varphi_0}^{K_0}$ is an isomorphism. We conclude (π, V) is equivalent to $\mathrm{ind}_B^G(\chi_{\lambda+1} \otimes 1)$. $\square$

4.7 Injectivity from $\mathrm{ind}_{K_0}^G\sigma/(T-\lambda)$ to principal series $\mathrm{ind}_B^G\varepsilon$: $\lambda \neq 0$

Let σ be an irreducible smooth representation of K_0, and ε be a character of the Borel subgroup B. In section 3.5, we have shown that the space $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma, \mathrm{ind}_B^G\varepsilon)$ is at most one-dimensional, and it is non-zero if and only if the restriction of ε to $B \cap K_0$ (i.e., ε_0 in the notation of 3.5) is equal to χ_σ^s. From now on, assume it is in this case. We have also defined a non-zero G-morphism $P_{v_0',1}$ in $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G\sigma, \mathrm{ind}_B^G\varepsilon)$, where $v_0' = \beta v_0$, and v_0 is a non-zero fixed vector in σ^{I_1}. By Proposition 3.24, $P_{v_0',1}$ will factor through the quotient $\mathrm{ind}_{K_0}^G\sigma/(T-c_\varepsilon)$, where c_ε is the value discovered in Proposition 3.24. Denote also by $P_{v_0',1}$ the reduced morphism from $\mathrm{ind}_{K_0}^G\sigma/(T-c_\varepsilon)$ to $\mathrm{ind}_B^G\varepsilon$.

The main result of this section is

Proposition 4.24. *When* $\dim\sigma > 1$ *and* $c_\varepsilon \neq 0$, $P_{v_0',1}$ *is injective.*

Proof. In the argument of Proposition 3.24, we have determined c_ε explicitly, which only depends on the character ε. Denote c_ε by λ. Under the assumption $\lambda \neq 0$, we proceed from Proposition 3.13 and have the following lemma, where we take the antecedent (Definition 3.31) into account.

Lemma 4.25. *$T[n_0\alpha^k,v]-\lambda[n_0\alpha^k,v]$*

$$= A[n_0\alpha^k,v]-\lambda[n_0\alpha^k,v]+\lambda^{-1}\sum_{y\in L_1\backslash\{0\},u_y=n(0,\varpi_E y)}(\lambda\cdot\mathfrak{f}_{0,u_y}-A\cdot\mathfrak{f}_{0,u_y})$$

$$+\lambda^{-1}\sum_{u\in(N_0\backslash N_1)/N_2}(\lambda\cdot\mathfrak{f}_{1,u}-A\cdot\mathfrak{f}_{1,u})\ +\lambda^{-1}\sum_{u_y=n(0,\varpi_E y),\ y\in L_1}(\lambda\cdot\mathfrak{f}_{2,u_y}-A\cdot\mathfrak{f}_{2,u_y})$$

where :

$$\mathfrak{f}_{0,u_y}=[n_0\cdot n(0,\varpi_E^{-(2k+1)}y)\alpha^k,\ \chi_1(\bar{y}_0^{-1})L_{v_0,y}(v)v_0],$$

for $u_y=n(0,\varpi_E y),y\in L_1\setminus\{0\}$;

$$\mathfrak{f}_{1,u}=[n_0\cdot n(\varpi_E^{-k}x_1\bar{y}_1^{-1},\varpi_E^{-2k}y_1^{-1})\alpha^{k-1},\ \chi_1(\bar{y}_1^{-1})\chi_2(-\bar{y}_1y_1^{-1})L_{v_0,y}(v)v_0],$$

for $u=n(x_1,y_1)\in(N_0\setminus N_1)/N_2$;

$$\mathfrak{f}_{2,u_y}=[n_0n(0,\varpi_E^{1-2k}y)\alpha^{k-1},\ S_{v_0}(v)v_0],$$

for $u_y=n(0,\varpi_E y),y\in L_1$.

Proof. We begin with the formula in Proposition 3.13. For $u\in N_1/N_2$, $n_0\alpha^k u\alpha^{-1}=n_0\cdot\alpha^k u\alpha^{-k}\cdot\alpha^{k-1}$. For $u\in N_2$, $n_0\alpha^k\beta u\alpha^{-1}=n_0\alpha^{k+1}\beta l$ for some $l\in N_0$.

For $u\in(N_1\setminus N_2)/N_2$, written as $n(0,\varpi_E y_0)$, we have

$$n_0\alpha^k\beta u\alpha^{-1}=n_0n(0,\varpi_E^{-(2k+1)}y_0)\alpha^k\mathrm{diag}(\bar{y}_0^{-1},\ 1,\ y_0)n'(0,\varpi_E y_0^{-1}).$$

For $u\in(N_0\setminus N_1)/N_2$, written as $n(x_1,y_1)$, we then have

$$n_0\alpha^k\beta u\alpha^{-1}=n_0n(\varpi_E^{-k}x_1\bar{y}_1^{-1},\varpi_E^{-2k}y_1^{-1})\alpha^{k-1}\mathrm{diag}(\bar{y}_1^{-1},\ -\bar{y}_1y_1^{-1},\ y_1)l'$$

for some $l'\in N_1'$.

We note that $A[n\alpha^k,\ v_0]$ is zero by our definition. Then we obtain the required formula in the Lemma by combining the above calculations. ☐

Using the above Lemma repeatedly, we get the following Corollary (compare with (19) in the proof of Theorem 4.22).

Corollary 4.26. *For any $n_0\in N$, $k\in\mathbb{Z}$, and $v\in W$, we have*

$$A[n_0\alpha^k,\ v]-\ \lambda[n_0\alpha^k,\ v]\equiv\textstyle\sum_j\ f_j(A[n_j\alpha^{k-1},\ v_j]-\ \lambda[n_j\alpha^{k-1},\ v_j])$$

for some $f_j\in\tilde{E}$ and some vectors $v_j\in W$. The elements n_j are all in N, satisfying that the vertices $n_j\alpha^{k-1}v_0$ are distinct from each other. Here the congruence is taken modulo $(T-\lambda)$.

Proof. Apply Lemma 4.25 to the terms where y goes through $(N_1\setminus N_2)/N_2$, observing that $L_{v_0,y}(v_0)=0$. ☐

After the previous preparation, we follow the process of Theorem 4.23 to prove $P_{v_0',1}$ is indeed injective.

Let $c \in S(G,\sigma)$ such that $P_{v_0',1}(c)=0$. We write c as $\sum_{j\in S}[n_j\alpha^j,u_j]$, where $n_j\in N$, $u_j\in W$. Let s be $\frac{1}{2}\min_{j\in S}\{h(n_j\alpha^j\mathbf{v}_0)\}-1$, and assume all the vertices $n_j\alpha^j\mathbf{v}_0$ are under $\mathbf{v}_{2r-2}$. Also, by setting some u_j to be 0, we may enlarge S so that the vertices $n_j\alpha^j\mathbf{v}_0$ go through all the vertices strictly under $\mathbf{v}_{2r}$ and with height at least $2s$. Using the following identity

$$[n\alpha^k,u]=\lambda^{-1}A[n\alpha^k,u]-\lambda^{-1}(A[n\alpha^k,u]-\lambda[n\alpha^k,u]),$$

we rewrite c as:

$$c=P\cdot[n\alpha^r,v_0']+\sum_{s<j<r}P_j\cdot(A[n_j\alpha^j,u_j]-\lambda[n_j\alpha^j,u_j]), \tag{29}$$

where P,P_j are some constants in $\tilde{E}$.

Combining the above equation with Corollary 4.26, we obtain

$$c\equiv P\cdot[n\alpha^r,v_0']+\textstyle\sum_{n\in N_{-2r}/N_{-2s}}P_n\cdot(A[n\alpha^s,v_0]-\lambda[n\alpha^s,v_0]).$$

By definition of antecedent, $A[n\alpha^s,v_0]=0$ for all n.

We recall we are in the case of $\dim\sigma>1$, the constant $\lambda_{\beta,\sigma}$ vanishes. Hence the function $\mathfrak{f}_0$, which is $P_{v_0',1}[Id,v_0]$, is 0 at Id and 1 at β. In our former notations, it is then just g_2. For simplicity, we would like to use $\varphi_2=1_{N_0^*}$, which corresponds to g_2.

We compute first $P_{v_0',1}([n\alpha^r,v_0'])=n\alpha^r\beta\mathfrak{f}_0=n\alpha^r\beta\varphi_2$, for which we understand it has non-compact support by pulling-back (also see Remark 4.7). Secondly, we compute $P_{v_0',1}([n\alpha^s,v_0])=n\alpha^s\mathfrak{f}_0=n\alpha^s 1_{N_0^*}=\varepsilon(\alpha)^{-s}1_{N^*_{-2s}\cdot(-x,\bar{y})}$ for $n=n(x,y)\in N_{-2r}/N_{-2s}$ which is compactly supported. Hence, we conclude that $P=0$. For the remaining terms, their supports $N^*_{-2s}\cdot(-x,\bar{y})$ are disjoint when $n=n(x,y)$ goes through N_{-2r}/N_{-2s}. We then conclude all the P_n are 0. We therefore have shown $c\in(T-\lambda)$. In all, the injectivity of $P_{v_0',1}$ is shown. ☐

4.8 Proof of (c) of (2) of Theorem 4.1

Proposition 4.27. *We have an isomorphism of G-representations:*

$$\mathrm{ind}_{K_0}^G St/(T)\cong\mathrm{ind}_B^G 1$$

Proof. We fix a non-zero vector v_0 in St^{I_1}. Now from section 3.5, the G-morphism $P_{v_0',1}\in\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G St,\ \mathrm{ind}_B^G 1)$ is well-defined.

The image of $P_{v_0',1}$ is generated by the function $P_{v_0',1}([Id, v_0]) = \mathfrak{f}_0$, which is just g_2 in our former notation. As g_2 is clearly not fixed by K_0, $P_{v_0',1}$ must be surjective due to the fact that $\mathrm{ind}_B^G 1$ has trivial character as the unique proper subrepresentation (see [Abd11]). From Proposition 3.24, $P_{v_0',1}$ is killed by T; actually one can simply check $P_{v_0',1}(T[Id, v_0]) = 0$. It suffices to prove $P_{v_0',1}$ is injective. However, it seems the strategy used in section 4.7 does not work when $\lambda = 0$.

We note that $P_{v_0',1}$ is surjective when restricted to the subspace of I_1-invariants. In fact, one can check $P_{v_0',1}(f_1) = g_1$, which is reduced to check $\sum_{u\in N_1'/N_2'} u\alpha\beta \cdot g_2 = g_1$.

We choose a proper character η of k_E^1, so that $\eta(-1) \neq 1$. Let $\sigma_1 = \eta \circ \det \otimes St$. The I_1-invariants of σ_1 are generated by v_0, on which I acts as character χ_{σ_1}. Hence, we may use the same notation $P_{v_0',1}$ as the non-zero G-morphism in $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma_1, \mathrm{ind}_B^G \eta\circ\det)$. In the same manner as we have just done, $P_{v_0',1}$ is surjective. By Proposition 3.24, $P_{v_0',1}$ factors through the quotient $\mathrm{ind}_{K_0}^G \sigma_1/(T' - (1-\eta(-1)))$. As $1-\eta(-1)) \neq 0$ and $\dim \sigma_1 > 1$, $P_{v_0',1}$ is injective, by Proposition 4.24. We are done, by applying Lemma 4.4. $\square$

We need the following analogue of Proposition 4.18, which is conjectured in [Abd11].

Proposition 4.28. *We have the following non-split short exact sequence:*

$$0 \to Triv \to \mathrm{ind}_{K_0}^G St/(T) \to Sp \to 0, \tag{30}$$

where St is inflation of the Steinberg representation of $\overline{G}(k_F)$ to K_0.

Proof. It is implied by Proposition 4.27, and the fact that $\mathrm{ind}_B^G 1$ is the non-split extension of Sp by trivial representation ([Abd11]). $\square$

We proceed to complete the proof of (c) of (2) of Theorem 4.1. In this case χ_σ factors through the determinant, i.e., $\chi_\sigma = \eta\circ\det$ for some character η of k_E^1, and $\lambda = 1 - \chi_1'(-1)$. From the theory of Carter-Lusztig ([KX12], (i) of Lemma 5.8), $\sigma \cong \eta\circ\det$ or $\sigma \cong \eta\circ\det\otimes St$, where St is the inflation of Steinberg representation of $\overline{G}(k_F)$ to K_0. In the first case $\mathrm{ind}_{K_0}^G \sigma \cong \eta\circ\det\otimes\mathrm{ind}_{K_0}^G 1$, and in the second case, $\mathrm{ind}_{K_0}^G \sigma \cong \eta\circ\det\otimes\mathrm{ind}_{K_0}^G St$. However, from Proposition 4.18 and Proposition 4.28, we conclude that $\eta\circ\det$ (resp. $\eta\circ\det\otimes St$) is the unique quotient of $\mathrm{ind}_{K_0}^G \eta\circ\det/(T-(1-\eta(-1)))$ (resp. $\mathrm{ind}_{K_0}^G \eta\circ\det\otimes St/(T-(1-\eta(-1)))$. Hence, we are done.

4.9 Proof of (b) of (2) of Theorem 4.1

We start to prove (b) of (2) in this section.

Before going into details, we recall a little more about the situation of (b). Under the assumption of (b), the principal series $\mathrm{ind}_B^G\varepsilon$ is irreducible nd there is a non-zero G-surjective morphism from $\mathrm{ind}_{K_0}^G\sigma$ to $\mathrm{ind}_B^G\varepsilon$, which factors through $\mathrm{ind}_{K_0}^G\sigma/(T-\lambda)$. We will prove that $\mathrm{ind}_{K_0}^G\sigma/(T-\lambda)$ is irreducible, which completes the argument that π is isomorphic to $\mathrm{ind}_B^G\varepsilon$.

For (b), we separate it into two cases:

Case 1 : σ is a character and $\lambda \neq 1-\chi'_{1,\sigma}(-1)$.

We repeat that $\mathrm{ind}_B^G\varepsilon$ is irreducible and is a quotient of $\mathrm{ind}_{K_0}^G\sigma/(T-\lambda)$. We reduce it to the unramified case which is already known. Write $\sigma = \eta\circ\det$ for a character η of k_E^1. Consider the principal series $\mathrm{ind}_B^G\varepsilon_1$, where $\varepsilon_1 \mid_{H_0} = (\eta^{-1}\circ\det)\cdot\varepsilon\mid_{H_0}$, $\varepsilon_1(\alpha)=\varepsilon(\alpha)$. Hence, $\mathrm{ind}_B^G\varepsilon_1$ is a quotient of $\mathrm{ind}_{K_0}^G 1/(T-\lambda_1)$, where $\lambda_1=\lambda+\chi'_{1,\sigma}(-1)-1$, by Proposition 3.24. The assumption on λ is translated into that $\lambda_1\neq 0,-1$. Hence, from the argument of Theorem 4.23 (2) (a), we have shown $\mathrm{ind}_{K_0}^G 1/(T-\lambda_1)\cong \mathrm{ind}_B^G(\chi_{\lambda_1+1}\otimes 1)$. We are done in this special case by twisting the character $\eta\circ\det$ back, applying Lemma 4.4.

Case 2 : The remaining cases of (b). Recall $\varepsilon(\alpha)\neq 0$ by our definition. In this case $\dim\sigma>1$ and the principal series $\mathrm{ind}_B^G\,\varepsilon$ is indeed irreducible by Theorem 4.9.

Subcase 1 : $\chi'_{1,\sigma}(-1)=1$

In the case that $\chi'_{1,\sigma}(-1)=1$, we are already done, as now the assumption of *Case 2* satisfies the conditions of Proposition 4.24.

Subcase 2 : $\chi'_{1,\sigma}(-1)=-1$

Choose a proper character η of k_E^1, so that $\chi'_{1,\sigma_1}(-1)=1$, where $\sigma_1=\eta\circ\det\otimes\sigma$. There is then a non-zero G-morphism from the compact induction $\mathrm{ind}_{K_0}^G\sigma_1$ to the principal series $\mathrm{ind}_B^G\varepsilon_1$, where ε_1 is the character of B: $\varepsilon_1\mid_{H_0}=\varepsilon\mid_{H_0}\cdot(\eta\circ\det)$, and $\varepsilon_1(\alpha)=\varepsilon(\alpha)$. By Proposition 3.24, such a G-morphism factors through $\mathrm{ind}_{K_0}^G\sigma_1/(T'-\lambda_1)$, where λ_1 is equal to $\lambda-2$ and is non-zero by the assumption on λ in this case.

Now, we can apply Proposition 4.24; as a result, we conclude that

$$\mathrm{ind}_{K_0}^G\sigma_1/(T'-\lambda_1)\cong\mathrm{ind}_B^G\varepsilon_1.$$

Finally, we twist both sides of the above isomorphism by the character $\eta^{-1}\circ\det$, using Lemma 4.4.

5 Canonical diagrams and finite presentation

This chapter is motivated by [Hu12] and [Sch12], as reflected from the title. In the case of $GL_2(\mathbf{Q}_p)$, all the irreducible smooth representations are finitely presented[9], which is a result of Barthel-Livné and Breuil. However, recent work of Hu [Hu12] and Schraen [Sch12] on $GL_2(F)$ has shown that supersingular representations are not finitely presented any more, when F is either a non-archimedean local field of positive characteristic or a quadratic extension of $\mathbf{Q}_p$.

The purpose of this chapter (and part of Appendix B) is to pursue some ideas underlying their work. Especially we follow Hu's canonical diagram [Hu12] closely and intend to arrive at some analogous results which are essentially used in Schraen's work [Sch12]. So far, we have achieved only part of the goal.

We now go into some details and explain the underlying motivations. Let σ be an irreducible smooth representation of K_0, and let π be a smooth G-quotient of $\mathrm{ind}_{K_0}^G\sigma$. In 3.7, $R_n^+(\sigma)$ $(n \geq 0)$ is defined as the subspace of $\mathrm{ind}_{K_0}^G\sigma$ which consists of functions supported in $K_0\alpha^n I$, and is I-stable. One has similar notation $R_n^-(\sigma)$ for $n \geq 0$, which consists of functions supported in $K_0\alpha^{-(n+1)}I$. There is then a natural I-decomposition of $\mathrm{ind}_{K_0}^G\sigma$:

$$\mathrm{ind}_{K_0}^G\sigma = I^+(\sigma) \oplus I^-(\sigma),$$

where $I^+(\sigma)$ (resp. $I^-(\sigma)$) is $\oplus_{n\geq 0}R_n^+(\sigma)$ (resp. $\oplus_{n\geq 0}R_n^-(\sigma)$). Denote by $I^+(\sigma,\pi)$ (resp. $I^{-1}(\sigma,\pi)$) the image of $I^+(\sigma)$ (resp. $I^-(\sigma)$) in π.

In the preliminary section 5.1, we have proved Proposition 5.3, following [Hu12] and combined with some result in previous chapters. It at least implies that the I-subrepresentation $I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$ of π is non-zero, if π is irreducible. We remark in the case of GL_2, this I-representation is the most basic ingredient in Hu's definition of canonical diagrams, and he has managed to show that it does not depend on the choice of the underlying weight σ, confirming the name of canonical. In our group $G = U(2,1)(E/F)$, we have not really pursued this seriously in this thesis, but only keep it in mind as a general guide.

In the second section 5.2 of this chapter, we aim to prove an analogue of a main result in [Hu12]. Assume π is an irreducible smooth representation of G and that it is the quotient of some compact induction $\mathrm{ind}_{K_0}^G\sigma$, via a projection θ. Generally speaking, to understand π it suffices to understand

[9] An irreducible smooth representation π is said to be finitely presented, if there is a non-zero G-morphism from some compact induction to π with the kernel finitely generated as a G-representation.

the corresponding kernel $R(\sigma,\pi)$. In view of Hu's idea, there is indeed a close relation between the I-representation $I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$ and $R(\sigma,\pi)$; in principle, $I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$ should inherit essential information from π itself. However, due to some technical difficulty, we have only been able to show the easy side, i.e., $I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$ is finite dimensional if $R(\sigma,\pi)$ is finitely generated (Proposition 5.6). Hence, π could not be finitely presented if one could show $I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$ is infinite dimensional. This is what Schraen has indeed proved for $GL_2(F)$ in [Sch12], when F is a quadratic extension of $\mathbf{Q}_p$. We also discuss informally, in the final part of this section, about the difficulty we have had in proving the converse (Remark 5.9).

In the section 5.3, we have mainly arrived at an analogue (Lemma 5.17) of a major technical result of Hu on $GL_2(F)$ ([Hu12], (i) of Proposition 4.11). It says that any N_0-invariant of $I^+(\sigma,\pi)$ is annihilated by some polynomial of S, where S is a canonically defined I_1-linear endomorphism of π^{N_0}, for an irreducible smooth supersingular representation π. However, at present it is not clear to us how to find interesting applications of this result.

In the section 5.4, using an argument of Paškūnas, we record a formal sufficient condition (Proposition 5.20) for the restriction to the Borel subgroup of an irreducible smooth representation to remain irreducible. However, in contrast to the case of GL_2, we don't expect the condition holds in general for supersingular representations of G.[10]

In the last section 5.5, we carry out some computations on the tree of G and prove that the dimension of the N_0-invariants of the space $\triangle(k,\sigma) := (R^+_{k-1}(\sigma)\oplus R^+_k(\sigma)\oplus R^+_{k+1}(\sigma))/T_\sigma(R^+_k(\sigma))$ ($k\geq 1$) is no less than $p(p-1)$ (Corollary 5.31), when $F=\mathbf{Q}_p$. The purpose that we do such thing is two-fold. On one side, the I-space $\triangle(k,\sigma)$ could be imagined as a finite piece of $\mathrm{ind}^G_{K_0}\sigma/(T_\sigma)$. In view of that, it is reasonable to believe one should get some interesting information of $\mathrm{ind}^G_{K_0}\sigma/(T_\sigma)$ by piecing together that of $\triangle(k,\sigma)$. On the other side, in the argument of a major result of Schraen ([Sch12], Proposition 13), a key point is reduced to checking the connecting homomorphism of some cohomology groups is not injective. The lower bound proved in this section could be used to verify a similar non-injective result (Remark 5.33). Hopefully, our result will also play a similar role in further considerations.

5.1 Preliminary results

We start by recalling some notations introduced in 3.7.

[10] Paškūnas proved in [Paš07] that a similar condition holds for supersingular representations of $GL_2(F)$.

Let σ be an irreducible smooth representation of K_0. We have set

$$R_n^+(\sigma) = [N_0\alpha^{-n}, \sigma], n \geq 0;\ R_{n-1}^-(\sigma) = [N_1'\alpha^n, \sigma], n \geq 1;$$
$$R_0(\sigma) = R_0^+(\sigma),\ R_n(\sigma) = R_n^+(\sigma) \oplus R_{n-1}^-(\sigma), n \geq 1.$$

We also set

$$I^+(\sigma) = \oplus_{n\geq 0} R_n^+(\sigma),\ I^-(\sigma) = \oplus_{n\geq 0} R_n^-(\sigma).$$

There is an I-decomposition

$$\mathrm{ind}_{K_0}^G \sigma = I^+(\sigma) \oplus I^-(\sigma)$$

Any $f \in \mathrm{ind}_{K_0}^G \sigma$ is therefore uniquely written as $f^+ + f^-$, for some $f^+ \in I^+(\sigma), f^- \in I^-(\sigma)$.

For $u \in \overline{U}$, let $[u]$ be a chosen element in N_0, satisfying that the reduction of $[u]$ is u. In the following, we will usually take $[u]$ as $n(x, y)$, for $(x, y) \in L_2$.

Lemma 5.1. *For $n \geq 1$, the space $R_n(\sigma)$, as a K_0-subrepresentation of $\mathrm{ind}_{K_0}^G \sigma$, is generated by $R_{n-1}^-(\sigma)$.*

Proof. Indeed, we have the following

$$R_n^+(\sigma) = \oplus_{k\in\overline{U}} [k]\beta R_{n-1}^-(\sigma),$$

which is directly from calculation. □

Recall we have shown in Proposition 3.37 that

$$T(R_n^-(\sigma)) \subseteq R_{n-1}^-(\sigma) \oplus R_n^-(\sigma) \oplus R_{n+1}^-(\sigma),$$

and $T\mid_{R_n^-(\sigma)}$ is the sum of I-morphism $T^- \mid_{R_n^-(\sigma)}:\ R_n^-(\sigma) \twoheadrightarrow R_{n-1}^-(\sigma)$ and the injective I-morphism $T^+\mid_{R_n^-(\sigma)}:\ R_n^-(\sigma) \to R_n^-(\sigma) \oplus R_{n+1}^-(\sigma)$ from Corollary 3.39.

Lemma 5.2. *Let $k \geq 0$, $f \in \oplus_{n\geq k} R_n^-(\sigma)$, and $P(x)$ any polynomial of degree at least one. Then there is $f' \in \oplus_{n\geq k+1} R_n^-(\sigma)$, depending on f and $P(x)$, such that*

$$f - f' \in P(T)(\oplus_{n\geq k+1} R_n^-(\sigma)).$$

Proof. One can write $P(x) = (x - \lambda)P_1(x)$ for some polynomial $P_1(x)$ of degree strictly smaller that of $P(x)$, and for some $\lambda \in \tilde{E}$. By the comment before the Lemma, we find some $f_1 \in \oplus_{n\geq k+1} R_n^-(\sigma)$, such that $T^-(f_1) = f$. If $P_1(x)$ is a constant, then the function $-T^+(f_1) + \lambda f_1$ is as desired. If not, we do induction on the degree of $P(x)$. Then, we are given $f_2, f_3 \in \oplus_{n\geq k+2} R_n^-(\sigma)$, such that $f_1 - f_2 = P_1(T)(f_3)$. Then, one can check the function

$$-T^{+}(f_1)+\lambda f_1+(T-\lambda)f_2$$

lies in $\oplus_{n\geq k+1}R_n^{-}(\sigma)$ and satisfies the requirement. $\square$

Let π be a G-quotient of the compact induction $\mathrm{ind}_{K_0}^{G}\sigma$. Denote by $R_n(\sigma,\pi)$ (resp. $R_n^{+}(\sigma,\pi), R_n^{-}(\sigma,\pi)$) the image of $R_n(\sigma)$ (resp. $R_n^{+}(\sigma), R_n^{+}(\sigma)$) in π. Denote by $\bar{f}$ the image of f in π, for a $f\in \mathrm{ind}_{K_0}^{G}\sigma$. Similarly, $I^{+}(\sigma,\pi)$ (resp. $I^{-}(\sigma,\pi)$) is the image of $I^{+}(\sigma)$ (resp. $I^{-}(\sigma)$) in π.

Proposition 5.3. *Assume π is an irreducible smooth representation and a G-quotient of $\mathrm{ind}_{K_0}^{G}\sigma$, and v_0 is a non-zero vector in σ^{I_1}. Then*
(1). $\overline{[Id,v_0]}\in\sum_{n\geq 0}R_n^{-}(\sigma,\pi)$;
(2). $R_0(\sigma,\pi)\subset\sum_{n\geq 1}R_n(\sigma,\pi)$.

We note (1) in particular implies $I^{+}(\sigma,\pi)\cap I^{-}(\sigma,\pi)\neq 0$.

Proof. (1). From Corollary 4.16, we see the given G-surjective map $\mathrm{ind}_{K_0}^{G}\sigma\twoheadrightarrow\pi$ will factor as

$$\mathrm{ind}_{K_0}^{G}\sigma\twoheadrightarrow\mathrm{ind}_{K_0}^{G}\sigma/P(T)\twoheadrightarrow\pi$$

for some polynomial $P(x)$ of degree greater than zero. Then the following claim will finish the proof of (1):

$$[Id,v_0]\in P(T)(\mathrm{ind}_{K_0}^{G}\sigma)+\oplus_{n\geq 0}R_n^{-}(\sigma). \tag{31}$$

We pick a root λ of $P(x)$ and write $P(x)=(x-\lambda)P_1(x)$ for some polynomial $P_1(x)$. Let f be the function $[\alpha,v_0]\in R_0^{-}(\sigma)$. A little calculation based on Proposition 3.41 shows that

$$(T-\lambda)f=[Id,v_0]+f_1 \tag{32}$$

for some $f_1\in\oplus_{n\geq 0}R_n^{-}(\sigma)$. If $P_1(x)$ is a constant, then the preceding identity (32) already gives us (31). Otherwise, using Lemma 5.2, we find some $f_2\in\oplus_{n\geq 1}R_n^{-}(\sigma)$ such that

$$f-f_2\in P_1(T)(\oplus_{n\geq 1}R_n^{-}(\sigma)),$$

which gives that $[Id,v_0]=(T-\lambda)f_2-f_1+P(T)f'$ for some $f'\in\oplus_{n\geq 1}R_n^{-}(\sigma)$, as desired for (31). We are done for (1) .

(2). The second part indeed follows from (1) directly, as one notes that $R_0(\sigma,\pi)$ is generated by $\overline{[Id,v_0]}$ as a K_0-representation. $\square$

Let ϕ_σ be the following I-morphism:

$$\phi_\sigma:\mathrm{ind}_{K_0}^{G}\sigma\twoheadrightarrow I^{-}(\sigma)\twoheadrightarrow I^{-}(\sigma,\pi)\hookrightarrow\pi,$$

where the first surjection on the left is the natural projection from $\mathrm{ind}_{K_0}^G\sigma$ to $I^-(\sigma)$.

Then, one has

Lemma 5.4. *$I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$ is the image of $R(\sigma,\pi)$ under ϕ_σ.*

Proof. This is indeed a formal result of Y.Hu, Lemma 3.11 in [Hu12]. □

Recall we have the coset decomposition:

$$K_0=\cup_{u\in N_0/N_1}[u]\beta I\cup I$$

Now we apply Lemma 2.10 in [Hu12]: take M to be the I-representation $I^-(\sigma,\pi)$, which generates π as a K_0 representation by Lemma 5.1. Consider the induced representation $W=\mathrm{Ind}_I^{K_0}M$, and from Frobenius reciprocity we are then given a surjective K_0-morphism $Pr:\mathrm{Ind}_I^{K_0}M\twoheadrightarrow\pi$, explicitly sending $[g,v]$ in $\mathrm{Ind}_I^{K_0}M$ to gv in π. Denote by $W_1(\sigma,\pi)$ be the kernel of Pr. As an I-representation, the following decomposition holds:

$$W=M\oplus W^+.$$

The underlying space of W^+ is generated by

$$\{[u\beta,v]:\ u=n(x,y),(x,y)\in L_2,v\in M\}.$$

Denote by Q^+ the image of W^+ in π. Then, from Lemma 5.1 again, we see it is just $\sum_{n\geq 1}R_n^+(\sigma,\pi)$. Lemma 2.10 of [Hu12] gives

$$W_1(\sigma,\pi)\subseteq\mathrm{Ind}_I^{K_0}(I^-(\sigma,\pi)\cap\textstyle\sum_{n\geq 1}R_n^+(\sigma,\pi))$$

Lemma 5.5. *Assume we have an identity in π of the following form:*

$$\textstyle\sum_{u=n(x,y),(x,y)\in L_2}u\beta v_u+v'=0$$

for some $v_u,v'\in I^-(\sigma,\pi)$. Then all the vectors v_u,v' lie in $I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$.

Proof. This is implied by the preceding remarks. □

5.2 An equivalent criteria for finiteness of $R(\sigma,\pi)$

Proposition 5.6. *Let π be an irreducible smooth representation of G and is a G-quotient of $\mathrm{ind}_{K_0}^G\sigma$. Let $R(\sigma,\pi)$ be the corresponding kernel. Then the following condition (2) implies (1) :*

(1). $I^+(\sigma,\pi)\cap I^-(\sigma,\pi)$ is of finite dimension ;

(2). $R(\sigma,\pi)$ is of finite type, as a $\tilde{E}[G]$-module.

Proof. Assume $\{f_1, f_2, \cdots, f_l\}$ is a finite set in $R(\sigma,\pi)$ which generates it over $\tilde{E}[G]$. For a large enough $m \geq 1$, all the f_i lie in $\oplus_{0\leq k\leq m} R_k(\sigma)$. Let M be the image of $\oplus_{0\leq k\leq m} R_k(\sigma)$ in π. From Lemma 5.4, we only need to show $\phi_\sigma(gf_i) \in M$ for all $g \in G$, as M is of finite dimension. Of course, it is the case for $g \in K_0$, as $\oplus_{0\leq k\leq m} R_n(\sigma)$ is stable under K_0. We are then reduced to the following simple lemma:

Lemma 5.7. *For any $n \geq 1$, $\phi_\sigma(\alpha^n f_i) \in M$.*

Proof. This results from some simple calculations. For $n \geq 1, \alpha^n f \in I^-(\sigma)$ when $f \in I^-(\sigma)$. Suppose k is an integer such that $0 \leq k \leq m$; then, for $v \in \sigma, n(x,y) \in N_0$, we have

$$[\alpha^n \cdot n(x,y) \cdot \alpha^{-k}, v]$$

$$\begin{cases} = [\alpha^n \cdot n(x,y) \cdot \alpha^{-n} \cdot \alpha^{-(k-n)}, v] \in I^+(\sigma), & \text{for } y \in \mathfrak{p}_E^{2n}, n \leq k; \\ \in I^-(\sigma), & \text{for } y \in \mathfrak{o}_E \setminus \mathfrak{p}_E^{2n}, n \leq k; \\ \in I^-(\sigma), & n > k. \end{cases}$$

Hence, we can conclude, for $f \in \oplus_{0\leq l\leq k} R_l(\sigma)$, when $n > k$, we have $(\alpha^n f^+)^+ = 0$, which gives $(\alpha^n f)^+ = 0$. When $n \leq k$, we also have $(\alpha^n f)^+ \in \oplus_{0\leq l\leq k} R_l(\sigma)$. For $f_i \in R(\sigma,\pi) \cap \oplus_{0\leq k\leq m} R_k(\sigma)$, we see $\phi_\sigma(\alpha^n f_i) = -\overline{(\alpha^n f_i)^+} \in M$ immediately. □

Remark 5.8. *We have indeed shown that $(\alpha^n f)^+ \in \oplus_{0\leq k\leq m} R_k(\sigma)$, for any $f \in \oplus_{0\leq k\leq m} R_k(\sigma)$ and any $n \geq 1$. In particular, $\phi_\sigma(\alpha^n f) \in M$, for any $f \in R(\sigma,\pi) \cap \oplus_{0\leq k\leq m} R_k(\sigma)$ and any $n \geq 1$.*

We turn to complete the proof from (2) to (1). As the $\oplus_{0\leq k\leq m} R_k(\sigma)$ is stable under K_0, we only need to show: if $f \in \oplus_{0\leq k\leq m} R_k(\sigma) \cap R(\sigma,\pi)$, then for any $n \geq 1, g \in K_0$, $\phi_\sigma(g\alpha^n f) \in M$. Clearly, by Remark 5.8, the claim is true if $g \in I$, as M is K_0-stable and ϕ_σ is I-map. Assume $g \in I\beta I$. Recall that $g\alpha^n f = g(\alpha^n f)^+ + g(\alpha^n f)^-$ and $\phi_\sigma(g\alpha^n f) = \overline{(g\alpha^n f)^-} = -\overline{(g\alpha^n f)^+}$. We only need to consider the case that $g = \beta$. But, by definition, $\phi_\sigma(\beta\alpha^n f) = \overline{(\beta(\alpha^n f)^+)^-}$, and the claim results from Remark 5.8. We are done. □

Remark 5.9. *We fail to prove (1) also implies (2). A simple reason we are not able to carry out the strategy in [Hu12] is the lack of an element in G which exchanges $I^+(\sigma)$ and $I^-(\sigma)$, which is due to the fact that the normalizer of I is itself; in the case of GL_2, the normalizer of the standard*

Iwahori subgroup contains an extra element[11] *which is not in the maximal compact open subgroup, and this element plays a very crucial role in Hu's argument.*

5.3 On the N_0-invariants of $I^+(\sigma, \pi)$

Lemma 5.10. *Let π be a supersingular representation of G, and assume θ is a non-zero G-morphism in $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma, \pi)$ for some compactly induced representation $\mathrm{ind}_{K_0}^G \sigma$. Then, for large enough $k \geq 1$, we have*

$$\theta \circ T_\sigma^k = 0.$$

Proof. From Corollary 4.16, there is a polynomial $f(T_\sigma) \in \mathscr{H}(K_0, \sigma)$ such that $\theta \circ f(T_\sigma) = 0$. Assume $f(T_\sigma)$ is such a polynomial of minimal degree. As π is supersingular, any root of $f(T_\sigma)$ must be zero, by Theorem 4.1. $\square$

In view of the Hecke operator T_σ, we define the following S, viewed as an operator on the non-zero space π^{N_0} for any smooth representation π.

Definition 5.11. *For any $v \in \pi^{N_0}$, Sv is defined as*

$$Sv = \textstyle\sum_{u \in N_0/N_2} u\alpha^{-1} \cdot v.$$

Proposition 5.12. (1). *Sv is well-defined and lies in π^{N_0}.*

(2). *For any $h \in H_1$, we have $h \cdot Sv = S(hv)$.*

(3). *If further v is fixed by I_1, the same is true for Sv.*

Proof. (1) is clear from the definition of S, which is well-defined as $v \in \pi^{N_0}$. For the proof of (2), we note first that $h \cdot v \in \pi^{N_0}$, as H_1 normalizes N_0. Then,

$$S(hv) = \textstyle\sum_{u \in N_0/N_2} uh\alpha^{-1}v = h(\sum_{u \in N_0/N_2} (h^{-1}uh)\alpha^{-1}v)$$

which is just $h(Sv)$ as required.

For (3), one needs some calculation as follows: from the Iwahori decomposition and (1), (2) just proved, we are reduced to check, for $n'(x, y) \in N'_1$

$$n'(x, y)Sv = n'(x, y) \textstyle\sum_{u \in N_0/N_2} u\alpha^{-1}v = \sum_{u \in N_0/N_2} n(x_2, y_2)^{-1}\alpha^{-1}\alpha b'\alpha^{-1}v,$$

where, writing $u = n(x_1, y_1) \in N_0$, b' is a lower triangular matrix as follows:

$$\begin{pmatrix} \frac{1}{1+x_1x+\overline{y_1y}} & 0 & 0 \\ \frac{x-\overline{x_1y}}{1+\overline{xx_1}+yy_1} & \frac{1+xx_1+\overline{yy_1}}{1+\overline{xx_1}+yy_1} & 0 \\ y & yx_1 - \bar{x} & 1 + \overline{xx_1} + yy_1 \end{pmatrix},$$

[11] That is $\begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}$.

and x_2, y_2 are then respectively

$$\tfrac{\overline{y_1 x} - x_1}{1 + \overline{y_1 y} + x_1 x}, \ \tfrac{\bar{y}_1}{1 + \overline{y_1 y} + x_1 x}.$$

As v is fixed by I_1, $\alpha b' \alpha^{-1} v = v$. Now we only need to see $n(x_2, y_2)^{-1}$ goes through N_0/N_2 when u does so. This is indeed the case. $\square$

To use S in a more efficient way, we also involve another linear map S_1 from π^{N_0} to $\pi^{N_1'}$ for a smooth representation π of G.

Definition 5.13. *For any $v \in \pi^{N_0}$, $S_1 v$ is defined as*

$$S_1 v = \sum_{u' \in N_1'/N_2'} u' \beta \alpha^{-1} v.$$

Proposition 5.14. (1) *S_1 is well-defined and $S_1 v \in \pi^{N_1'}$ for $v \in \pi^{N_0}$.*

(2) *$S_1 \cdot hv = h^s \cdot S_1 v$, for a diagonal h in I.*

(3) *$S_1 v \in \pi^{I_1}$ if $v \in \pi^{I_1}$.*

Proof. Assertions in (1) and (2) are easily checked from the definition of S_1. Now a similar calculation to that of Proposition 5.12, combined with (1) and (2), confirms (3). $\square$

Lemma 5.15. *Let π be a smooth representation of G. Given $0 \neq v \in \pi^{I_1}$, such that I acts on v as a character, either $Sv = 0$ or $\langle K_0 \cdot Sv \rangle$ is an irreducible representation of K_0 of dimension bigger than 1.*

Proof. Assume $Sv \neq 0$. Then by definition of S and S_1, $v' = S_1 v$ is also non-zero. Consider the K_0-representation $\kappa = \langle K_0 \cdot v' \rangle$. As I acts on v by a character χ, I acts on v' by χ^s, from (2) of Proposition 5.14. Hence, from Frobenius reciprocity there is a surjective K_0-morphism from $\mathrm{Ind}_I^{K_0} \chi^s$ to κ, which sends φ_{χ^s} to v'. From the definition of S, one has

$$Sv = \sum_{u = n(x,y), (x,y) \in L_2} u \beta v';$$

As a result, $\langle K_0 \cdot Sv \rangle$ is the image of $\langle K_0 \cdot \sum_{u = n(x,y), (x,y) \in L_2} u \beta \varphi_{\chi^s} \rangle$, which is an irreducible representation of K_0 of dimension bigger than one, due to a general result of Carter-Lusztig, see (i) of Proposition 5.7 of [KX12]. Hence the assertion. $\square$

Corollary 5.16. *Assume π is a supersingular representation of G and v is a non-zero vector in π^{I_1}. Then there exists a family of $c_i \in \tilde{E}$ and integer $k \geq 0$ such that,*

$$\prod_i (S + c_i)^k S v = 0.$$

Proof. We assume first I acts on v as a character χ.

From Lemma 5.15, the K_0-representation σ generated by Sv is irreducible of dimension bigger than one, if $Sv \neq 0$.

Assume $Sv \neq 0$. We are then given a G-morphism θ in $\mathrm{Hom}_G(\mathrm{ind}_{K_0}^G \sigma, \pi)$, determined by $\theta([Id, Sv]) = Sv$. From Lemma 5.10 and (14), there is some constant $c \in \tilde{E}$ and some $k \geq 1$ such that

$$(S + c)^k Sv = 0,$$

and we are done in this special case.

As I/I_1 is an abelian group of finite order prime to p, for any non-zero $v \in \pi^{I_1}$, the I-representation $< I \cdot v >$ is just a sum of characters, therefore one may write v as a sum $\sum v_i$, where I acts on v_i by some character χ_i of I/I_1. We then apply the former process to each v_i, hence the result. □

Lemma 5.17. *Suppose π is a supersingular representation and a G-quotient of $\mathrm{ind}_{K_0}^G \sigma$. If $0 \neq v \in I^+(\sigma, \pi)$ is fixed by N_0, then there is a polynomial P of degree ≥ 1, such that:*

$$P(S)v = 0.$$

Proof. Based on Corollary 5.16, the result follows by an induction argument, due to Y. Hu [Hu12].

Denote by m_v the dimension of the I_1-representation $\langle I_1 \cdot v \rangle$, for $v \in I^+(\sigma, \pi)^{N_0}$.

When $m_v = 1$, i.e., v is fixed by I_1, the assertion is just the content of Corollary 5.16. Assume $m_v \geq 2$ and the Lemma is true for any $v' \in I^+(\sigma, \pi)^{N_0}$ such that $m_{v'} < m_v$. Then for any $h \in I_1 \cap H$, $m_{(h-1)v}$ is strictly smaller than m_v, from Lemma 4.12 of [Hu12]. By the induction hypothesis, there is a non-constant polynomial P_h such that

$$P_h(S)(h - 1)v = 0.$$

However, as $\langle (I_1 \cap H)v \rangle$ is of finite dimension, one could then choose a non-constant polynomial P' such that

$$P'(S)(h - 1)v = 0$$

for all $h \in I_1 \cap H$. This just says $h \cdot P'(S)v = P'(S)v$, by (2) of Proposition 5.12. In other words, by replacing v with $P'(S)v$, for some non-constant polynomial P', one may assume further that v is fixed $H \cap I_1$.

Next, v is fixed by N'_{2k+1} for some $k \geq 1$, as π is a smooth representation. Now, the same calculation used in Proposition 5.12 gives us that, for a $u' \in N'_{2k-1}$,

$$u' \cdot Sv = u' \sum_{u \in N_0/N_2} u\alpha^{-1}v = \sum_{u \in N_0/N_2} u_1\alpha^{-1}\alpha b'\alpha^{-1}v,$$

where u_1 goes through N_0/N_2 when u goes through N_0/N_2, and $\alpha b'\alpha^{-1}$ is lower triangular matrix in N'_{2k+1}, as $u' \in N'_{2k-1}$ (from the explicit description of b' in the argument of Proposition 5.12). Therefore, one concludes that Sv is fixed by N'_{2k-1} . Repeating the process for enough times, we have shown $S^k v$ is fixed by N'_1. Hence $S^k v$ is fixed by I_1, and the Lemma follows then by using Corollary 5.16 again. We are done. □

We record the following observation as a corollary, in which we assume π is smooth irreducible and that there is a G-morphism from a compact induction $\mathrm{ind}_{K_0}^G \sigma$ to π.

Corollary 5.18. *Let $0 \neq v \in I^+(\sigma, \pi)^{N_0}$, and $Sv = 0$. Then $S_1 v \in I^+(\sigma, \pi) \cap I^-(\sigma, \pi)$.*

Proof. By (1) of Proposition 5.14, $S_1 v$ is well-defined. As $v \in I^+(\sigma, \pi)$, a simple calculation shows that $S_1 v \in I^-(\sigma, \pi)$. Now from the assumption that $Sv = 0$, the result follows from Lemma 5.5. □

Conjecture 5.19. *Let π be an irreducible supersingular representation. Then one has the following inclusion:*

$$S_1(I^+(\sigma, \pi)^{N_0}) \subseteq \sum_{k \geq 0} S^k(I^+(\sigma, \pi) \cap I^-(\sigma, \pi))$$

5.4 Restriction to Borel subgroup

The following formal result, whose proof is due to V. Paškūnas([Paš07]), provides some evidence that the definition of S is reasonable.

Proposition 5.20. *Let π be an irreducible smooth representation of G. If, for any non-zero vector $w \in \pi$, there is a non-zero vector $v \in \pi^{I_1} \cap \langle B \cdot w \rangle$ such that*

$$Sv = 0,$$

then $\pi \mid_B$ is irreducible.

Proof. Let w be a non-zero vector in π. As π is a smooth representation, there exists a $k \geq 0$ such that w is fixed by N'_{2k+1}. Hence, $w_1 = \alpha^{-k} w$ is fixed by N'_1. From the Iwahori decomposition $I_1 = (I_1 \cap B) \cdot N'_1$, we see

$$\langle I_1 \cdot w_1 \rangle = \langle (I_1 \cap B) \cdot w_1 \rangle.$$

As I_1 is a pro-p group, the I_1-invariants of above smooth representation is non-zero. We have shown $\pi^{I_1} \cap \langle B \cdot w \rangle \neq 0$.

We record the following lemma, which makes the whole thing more apparent.

Lemma 5.21. *If $Sv = 0$, then $\beta v \in \langle B \cdot v \rangle$.*

Proof. Assume $Sv = 0$. Hence, we get

$$v = \alpha \cdot \textstyle\sum_{u \in (N_0 \backslash N_2)/N_2} u\alpha^{-1} v$$

Therefore, $\beta v = \sum_{(N_0 \backslash N_1)/N_2} \beta\alpha u \alpha^{-1} v + \sum_{(N_1 \backslash N_2)/N_2} \beta \alpha u \alpha^{-1} v$. Repeatedly using of Lemma 1.2, one see both sums in the former equation lie in $\langle B \cdot v \rangle$. $\square$

We continue with proof of Proposition 5.20. Choose $0 \neq v \in \pi^{I_1} \cap \langle B \cdot w \rangle$ such that $Sv = 0$. The above lemma says $\beta v \in \langle B \cdot v \rangle$. As π is irreducible, from the decomposition $G = BI_1 \cup B\beta I_1$, we see

$$\pi = \langle G \cdot v \rangle = \langle B \cdot v \rangle \subseteq \langle B \cdot w \rangle.$$

Hence, we have shown $\pi = \langle B \cdot w \rangle$ for any $w \in \pi$. We are done. $\square$

Remark 5.22. *Clearly from the argument, the above proposition still holds if the condition that $Sv = 0$ is replaced by $\beta v \in \langle B \cdot v \rangle$.*

Remark 5.23. *Of course, the condition in Proposition 5.20 is only sufficient. For example, one may check easily $Sp \mid_B$ is irreducible : We have shown $\bar{g}_1$ generates Sp^{I_1}. In fact, a further look of the identity in (2) of Proposition 4.14 gives immediately that $\beta \bar{g}_1 \in \langle B \cdot \bar{g}_1 \rangle$, hence the claim by last remark. But one can check $S \cdot \bar{g}_1 \neq 0$.*

Certainly, the most interesting case is to check what happens for supersingular representations π, which is also the goal of this section. However, in view of Proposition 5.20, we only have the following embarrassing result:

Corollary 5.24. *Let π be a supersingular representation of G. If all the underlying weights σ of π with $\dim \sigma > 1$ satisfy $\chi_\sigma \neq \chi_\sigma^s$, then $\pi \mid_B$ is irreducible.*

5.5 Estimation of N_0-invariants of $R^+_{k-1} \oplus R^+_k \oplus R^+_{k+1}/T(R^+_k)$ for $k \geq 1$ when $F = \mathbf{Q}_p$

It is very likely that $\mathrm{ind}_{K_0}^G \sigma/(T_\sigma)$ is non-admissible, and one goal of the section is to provide some evidence on that. We carry out some local computations on the tree, some of which work for any field. But for simplicity, in the main result Proposition 5.27 we pursue it under the assumption that $F = \mathbf{Q}_p$. See Remark 5.33 for more details that how it would be applied.

The group N_k is non-commutative, and we denote its center by C_{N_k}.

Lemma 5.25. *The $u(y)$-translations of $u_{k,v}$ consist of a basis of the C_{N_0}-invariants of $R^+_k(\sigma)$, where y goes through $\mathfrak{o}_E/\mathfrak{p}_E^k$, v goes through a basis of $\sigma^{C_{N_k}}$, and $u_{k,v}$ is the following function:*

$$u_{k,v} = \textstyle\sum_{u \in C_{N_0}/C_{N_{2k}}} [u\alpha^{-k}, v].$$

Proof. First recall the double coset decomposition

$$K_0\alpha^k N_0 = \cup_{y \in \mathfrak{o}_E/\mathfrak{p}_E^k} K_0\alpha^k u(y) C_{N_0}.$$

It is clear that the functions $u_{k,v}$ and their translations by $u(y)$ are C_{N_0}-invariant and linearly independent, and we only need to show any C_{N_0}-invariant function in $R^+_k(\sigma)$ is a linear combination of them.

Let f be a C_{N_0}-invariant function in $R^+_k(\sigma)$, supported on $K_0\alpha^k C_{N_0}$. The value of f at $\alpha^k u$ for $u \in C_{N_0}$ is then a C_{N_0}-invariant vector in σ, and the Lemma follows. $\square$

Fix a non-zero vector $v_0 \in \sigma^{I_1}$, and write u_k for u_{k,v_0}. Denote by $D_k(\sigma)$ the subspace of $R^+_k(\sigma)$ which is generated by the $u(y)$-translation of u_k, for all $y \in \mathfrak{o}_E/\mathfrak{p}_E^k$. Recall that the T^+ respects the action of N_0, in particular it preserves C_{N_0}-invariants. Then, we have,

Proposition 5.26. $T^+(u_k) = \lambda_{\beta,\sigma} c_\sigma \cdot u_k + \sum_{y \in \mathfrak{p}_E^k/\mathfrak{p}_E^{k+1}} u(y) \cdot u_{k+1}$, *where* $u(y) = n(y, -\frac{y\bar{y}}{2})$, *and c_σ is described in the proof.*

Proof. This is from explicit calculations. As T is G-equivalent, we have from the argument of Proposition 3.37 for a $u \in C_{N_0}$

$$T^+([u\alpha^{-k}, v_0]) =$$
$$u\alpha^{-k} \textstyle\sum_{v \in N_0/N_2} [v\alpha^{-1}, v_0] + u\alpha^{-k}\lambda_{\beta,\sigma} \sum_{w \in (N_1' \setminus N_2')/N_2'} [w\alpha, v_0'],$$

where the first sum is in $R^+_{k+1}(\sigma)$ and the second sum is $R^+_k(\sigma)$. Hence, we have

$$T^+(u_k) = \sum_{v \in N_0/N_2} \sum_{u \in C_{N_0}/C_{N_{2k}}} [u\alpha^{-k}v\alpha^{-1}, v_0] + \sum_{w \in (N'_1 \backslash N'_2)/N'_2} \sum_{u \in C_{N_0}/C_{N_{2k}}} [u\alpha^{-k}w\alpha, v'_0].$$

When v goes through N_0/N_2, $v_1 = \alpha^{-k}v\alpha^k$ goes through N_{2k}/N_{2k+2}. We split v_1 into $v_2 \cdot u(y)$, where $v_2 \in C_{2k}/C_{2k+2}$ and $y \in \mathfrak{p}^k_E/\mathfrak{p}^{k+1}_E$. We are done for the second sum in the Proposition, by the definition of u_{k+1}.

Write $w = n'(0, \varpi_E y)$ where $y \in L^*_1$. We have by Lemma 1.2 that

$$\alpha^{-k}w\alpha^k = \beta n(0, \varpi_E^{-(2k-1)}y)\beta = n(0, \varpi_E^{2k-1}y^{-1})\alpha^{-(2k-1)}n'(0, \varpi_E^{2k-1}y^{-1})h_w\beta,$$

where $h_w = \mathrm{diag}(y, 1, \bar{y}^{-1})$. As $h \cdot v_0 = \chi_\sigma(h)v_0$, and $\beta v'_0 = v_0$, which is fixed I_1, we obtain

$$\sum_{w \in (N'_1 \backslash N'_2)/N'_2} \sum_{u \in C_{N_0}/C_{N_{2k}}} \chi_\sigma(h_w)[u \cdot n(0, \varpi_E^{2k-1}y^{-1})\alpha^{-k}, v_0].$$

By noting $n(0, \varpi_E^{2k-1}y^{-1}) \in C_{N_0}$, we finally get

$$\sum_{w \in (N'_1 \backslash N'_2)/N'_2} \chi_\sigma(h_w) \sum_{u \in C_{N_0}/C_{N_{2k}}} [u\alpha^{-k}, v_0]$$

as required, where the sum $\sum_{w \in (N'_1 \backslash N'_2)/N'_2} \chi_\sigma(h_w)$ is the constant c_σ. We are done. □

Based on last Proposition, we are led to the main result of this section.

Proposition 5.27. *When $F = \mathbf{Q}_p$, the subspace of the functions in $D_k(\sigma) \oplus D_{k+1}(\sigma)$ which are N_0-invariant in the quotient $R^+_k \oplus R^+_{k+1}/T^+(R^+_k)$ is at least of dimension $p(p-1)+1$.*

Proof. When $q = p$, the group N_0 is generated (topologically) by three elements, say $n(1, -\frac{1}{2}), n(\eta, -\frac{\eta\bar{\eta}}{2})$ and $n(0, \eta - \bar{\eta})$. Hence by restricting to C_{N_0}-invariant functions, we only need to consider the actions of first two elements.

Assume $f = \sum_{y \in \mathfrak{o}_E/\mathfrak{p}^k_E} l'_y u(y) \cdot u_k + \sum_{y \in \mathfrak{o}_E/\mathfrak{p}^{k+1}_E} l_y u(y) \cdot u_{k+1}$ is a function in $D_k(\sigma) \oplus D_{k+1}(\sigma)$ whose image in $R^+_k \oplus R^+_{k+1}/T^+(R^+_k)$ is N_0-invariant. Therefore, we are reduced to looking for functions $g_1, g_2 \in R^+_k$, such that

$$u(1) \cdot f - f = T^+ \cdot g_1,\ u(\eta) \cdot f - f = T^+ \cdot g_2.$$

We note that the existence of g_i implies that it must be a C_{N_0}-invariant function, as T^+ is injective. In the following we will show g_i could be chosen in $D_k(\sigma)$; as a result they are uniquely determined.

Now we could in principle apply Proposition 5.26 and compare the coefficients to solve out the involved parameters. However, as we don't really need to determine all of the solutions, we restrict to those satisfying

$$* \; l'_y = l'_{y'} \text{ and } l_y = l_{y'} \text{ if } y \equiv y' \bmod \mathfrak{p}_E.$$

We now re-write both side of the equation $u(1) \cdot f - f = T^+ \cdot g_1$ as

$$u(1) \cdot f - f = \textstyle\sum_{y \in \mathfrak{o}_E/\mathfrak{p}^k_E} (l'_{y-1} - l'_y) u(y) u_k + \sum_{y \in \mathfrak{o}_E/\mathfrak{p}^{k+1}_E} (l_{y-1} - l_y) u(y) u_{k+1}$$

$$T^+(\textstyle\sum_{y \in \mathfrak{o}_E/\mathfrak{p}^k_E} d_{1,y} u(y) u_k) =$$
$$\textstyle\sum_{y \in \mathfrak{o}_E/\mathfrak{p}^k_E} d_{1,y} u(y)(\lambda_{\beta,\sigma} c_\sigma u_k + \sum_{y_1 \in \mathfrak{p}^k_E/\mathfrak{p}^{k+1}_E} u(y_1) u_{k+1}) =$$
$$\textstyle\sum_{y \in \mathfrak{o}_E/\mathfrak{p}^k_E} \lambda_{\beta,\sigma} c_\sigma d_{1,y} u(y) u_k + \sum_{y \in \mathfrak{o}_E/\mathfrak{p}^{k+1}_E} d_{1,y} u(y) u_{k+1}.$$

In the above we note that $y + y_1$ goes through $\mathfrak{o}_E/\mathfrak{p}^{k+1}_E$ when y and y_1 go through respectively $\mathfrak{o}_E/\mathfrak{p}^k_E$ and $\mathfrak{p}^k_E/\mathfrak{p}^{k+1}_E$, and that $u(y)u(y_1) = u(y + y_1) \cdot c$ for some $c \in C_{N_0}$ (c depends on y and y_1). Hence, we are lead to

$$l'_{y-1} - l'_y = \lambda_{\beta,\sigma} c_\sigma d_{1,y}, \; y \in \mathfrak{o}_E/\mathfrak{p}^k_E. \tag{33}$$

and

$$l_{y-1} - l_y = d_{1,y}, \; y \in \mathfrak{o}_E/\mathfrak{p}^{k+1}_E. \tag{34}$$

From the second equation $u(\eta) \cdot f - f = T^+ \cdot g_2$, we have similarly

$$l'_{y-\eta} - l'_y = \lambda_{\beta,\sigma} c_\sigma d_{2,y}, \; y \in \mathfrak{o}_E/\mathfrak{p}^k_E. \tag{35}$$

and

$$l_{y-\eta} - l_y = d_{2,y}, \; y \in \mathfrak{o}_E/\mathfrak{p}^{k+1}_E. \tag{36}$$

Then under the assumption $*$, we are required to look those l'_y (resp. l_y) for $y \in \mathfrak{o}_E/\mathfrak{p}^k_E$ (resp. $y \in \mathfrak{o}_E/\mathfrak{p}^{k+1}_E$), which makes the families of equations in (33) and (35) (resp. (34) and (36)) compatible. We recall that in most cases $\lambda_{\beta,\sigma}$ and c_σ are just zero. In that situation, we simply take all the l'_y to be zero (i.e., we throw all the other non-trivial solutions). In the cases that $\lambda_{\beta,\sigma} c_\sigma \neq 0$, the assumption $*$ reduces (33) and (35) as

$$l'_{y-1} - l'_y = \lambda_{\beta,\sigma} c_\sigma d_{1,y}, \; y \in \mathfrak{o}_E/\mathfrak{p}_E. \tag{37}$$

and

$$l'_{y-\eta} - l'_y = \lambda_{\beta,\sigma} c_\sigma d_{2,y}, \; y \in \mathfrak{o}_E/\mathfrak{p}_E. \tag{38}$$

Now we identify η with a generator of k_E over k_F, hence we identify k_E with $k_F \oplus k_F \eta$.

Then the families of equations (37) and (38) could be re-written as,

$$l'_{a-1+b\eta} - l'_{a+b\eta} = \lambda_{\beta,\sigma} c_\sigma d_{1,a+b\eta}, \; a, b \in k_F. \tag{39}$$

$$l'_{a+(b-1)\eta} - l'_{a+b\eta} = \lambda_{\beta,\sigma} c_\sigma d_{2,a+b\eta}, \; a, b \in k_F. \tag{40}$$

In the families of (39) or (40), for a fixed b, we look at those p equations for which a goes through k_F. Immediately, we see the following conditions are necessary:

$$\sum_{a \in k_F} d_{1,a+b\eta} = 0, \sum_{a \in k_F} d_{2,a+b\eta} = 0. \tag{41}$$

We note for a fixed $b \in k_F$, the solutions $\{d_{i,a+b\eta}\}_{a \in k_F}$ in former equation consist of a linear space over k_F of dimension $p-1$. Then given a such (non-trivial) solution $\{d_{i,a+b\eta}\}_{a \in k_F}$, the solutions of those p equations (more precisely) are uniquely determined up to adding a one-dimensional vector space, as the underlying matrix of the coefficients is of rank $p-1$. We need to make the former description more precisely. By identifying k_F with $\mathbb{F}_p = \{i; 0 \leq i \leq p-1\}$, for a fixed j satisfying $0 \leq j \leq p-1$, we have from (39)

$$l'_{i+j\eta} = l'_{j\eta} - \lambda_{\beta,\sigma} c_\sigma \textstyle\sum_{0<i' \leq i} d_{1,i'+j\eta} \text{ for } i > 0;$$

Similarly, for a fixed i satisfying $0 \leq i \leq p-1$, we have from (40) that

$$l'_{i+j\eta} = l'_i - \lambda_{\beta,\sigma} c_\sigma \textstyle\sum_{0<j' \leq j} d_{2,i+j'\eta} \text{ for } j > 0.$$

Putting them together, we have uniformly for $ij \neq 0$ (the case that $ij = 0$ is already covered in last two equations) that,

$$l'_{i+j\eta} = l'_0 - \lambda_{\beta,\sigma} c_\sigma \sum_{0<j' \leq j} d_{2,j'\eta} - \lambda_{\beta,\sigma} c_\sigma \sum_{0<i' \leq i} d_{1,i'+j\eta}. \tag{42}$$

In summary, for a given family of $\{d_{1,a+b\eta},\ d_{2,a+b\eta}\}_{0 \leq a,b \leq p-1}$ satisfying condition (41), the solution $\{l'_{i+j\eta}\}_{0 \leq i,j \leq p-1}$ is uniquely determined by l'_0.

We repeat the whole process to treat the equations (34) and (36) and conclude similarly that for a given family of $\{d_{1,a+b\eta},\ d_{2,a+b\eta}\}_{0 \leq a,b \leq p-1}$ satisfying condition (41), the solution $\{l_{i+j\eta}\}_{0 \leq i,j \leq p-1}$ is uniquely determined by l_0.

Hence the dimension of the space of the solutions $\{l_{i+j\eta},\ l'_{i+j\eta};\ 0 \leq i,j \leq p-1\}$ is at least $p(p-1)+1$.

We are done for the proof of the Proposition. □

Corollary 5.28. *When $F = \mathbf{Q}_p$, σ is a character, the N_0-invariant of $R^+_k(\sigma) \oplus R^+_{k+1}(\sigma)/T^+(R^+_k(\sigma))$ is at least of dimension $p(p-1)$.*

Proof. We need to estimate those solutions described in Proposition 5.27 which lie in the image of T^+. Recall the solutions in 5.27 are all C_{N_0}-invariant, and as T^+ is injective we see the solutions with a form $T^+(f)$ for some $f \in R^+_k(\sigma)^{C_{N_0}}$ is at most one-dimensional, by comparing the support of the solution and that of $T^+ f$, using Lemma 5.25. □

In view of the above corollary, it is reasonable to hope its generalization holds, although currently we are not able to verify it:

Conjecture 5.29. *Those solutions described in Proposition 5.27 which become zero in $R^+_k(\sigma) \oplus R^+_{k+1}(\sigma)/T^+(R^+_k(\sigma))$ are at most one dimensional and the N_0-invariant of $R^+_k(\sigma) \oplus R^+_{k+1}(\sigma)/T^+(R^+_k(\sigma))$ is at least of dimension $p(p-1)$, for any irreducible smooth representation σ of K_0.*

Remark 5.30. *Assume Conjecture 5.29 holds. It is directly to check that the solutions from Proposition 5.27 are killed by T^-; as a result we see $f_{-(k-1)} + g_k + g_{k+1}$ is a function in $R^+_{k-1} \oplus R^+_k \oplus R^+_{k+1}$ whose image in the quotient $R^+_{k-1} \oplus R^+_k \oplus R^+_{k+1}/T(R^+_k)$ is N_0-invariant, for any function $g_k + g_{k+1}$ in $R^+_k \oplus R^+_{k+1}$ found in the proposition satisfying its image in $R^+_k \oplus R^+_{k+1}/T^+(R^+_k)$ is N_0-invariant. Such process is indeed injective, hence we have shown the following corollary.*

Corollary 5.31. *Assume Conjecture 5.29 holds. The dimension of N_0-invariants of $R^+_{k-1} \oplus R^+_k \oplus R^+_{k+1}/T(R^+_k)$ is bigger or equal than $p(p-1)$.*

Remark 5.32. *In the statement of Corollary 5.31, one may replace T by $T - \lambda$, for any $\lambda \in \tilde{E}$.*

Remark 5.33. *We briefly comment on how the above Proposition is expected to be used. We begin with the following short exact sequence of smooth N_0-representations induced by the Hecke operator T :*

$$0 \to R^+_1 \xrightarrow{T} R^+_0 \oplus R^+_1 \oplus R^+_2 \to \mathrm{Coker}(T\mid_{R^+_1}) \to 0$$

Then the derived long exact sequence will give rise to a $H^1(T)$ map :

$$H^1(T):\ H^1(R^+_1) \to H^1(R^+_0 \oplus R^+_1 \oplus R^+_2).$$

Then the result from last Corollary (combining Lemma 3.38) will guarantee $H^1(T)$ is not injective.

6 Appendix A: Coefficient systems and Diagrams

In this appendix, besides other things, we follow [Paš04] to prove some formal results which interpret G-equivariant coefficient systems on $\triangle$ in terms of diagrams, which is the content of Theorem 6.2.

We remind the readers that, the concept of equivariant coefficient systems on the Bruhat–Tits building of a p-adic connected reductive group is given by Schneider and Stuhler in [SS97], in which it is in the context of complex representations. However, the concept also works for fields of positive characteristic. In [Paš04], Paškūnas firstly used it to construct supersingular representations of $GL_2(F)$. Clearly, our presentation in this chapter is close to that in [Paš04]. It is believable that Theorem 6.2 should hold in much more general settings, if one has the right definition of diagrams.

6.1 Coefficient systems and Diagrams

Assume $\tilde{E}$ is an algebraically closed field of characteristic p. Let X_0 be the set of all vertices on the tree $\triangle$, and X_1 be the set of all edges on X. In this chapter, we will denote the two vertices $\mathbf{v}_0$ and $\mathbf{v}_1$ respectively by σ_0 and σ_1, and the edge $e_{0,1}$ by $\tau_{0,1}$. The stabilizers of $\sigma_0, \sigma_1, \tau_{0,1}$ are respectively $R(\sigma_0) = K_0$, $R(\sigma_1) = K_1$, $R(\tau_{0,1}) = I$.

Let $\mathcal{V} = (V_\sigma)_\sigma$ be a coefficient system on $\triangle$, i.e., for each simplex σ, V_σ is a given $\tilde{E}$-vector space; for each pair $\sigma \subseteq \tau$ of simplices, there is a given linear map r^σ_τ from V_τ to V_σ, satisfying $r^\sigma_\sigma = Id_\sigma$; for each $g \in G$ and each simplex σ, there is a given linear map g_σ from V_σ to $V_{g\sigma}$, which is compatible with the action of G on $\triangle$ and commutes the restriction maps in the obvious way. For each simplex σ, the stabilizer $R(\sigma)$ of σ acts linearly on V_σ. An G-equivariant coefficient system $(V_\sigma)_\sigma$ is a coefficient system, satisfying that the action of $R(\sigma)$ on V_σ is smooth, for any simplex σ.

Denote by $\mathcal{COEF}_G$ the category of G-equivariant coefficient systems on $\triangle$, with the natural morphisms. Before going into more details, we record the following useful fact.

Let $\mathcal{V} = (V_\sigma)_\sigma$ be a G-equivariant system, and let $\tau = (\sigma, \sigma')$ be an edge, then there is $g \in G$ such that, $\tau = g\tau_{0,1}$, in other words, $\sigma = g\sigma_0$, $\sigma' = g\sigma_1$. Without difficulty, we see

$$V_\sigma = g_{\sigma_0} V_{\sigma_0},\ V_{\sigma'} = g_{\sigma_1} V_{\sigma_1},\ V_\tau = g_{\tau_{0,1}} V_{\tau_{0,1}}.$$

From these translation relations, we have the following relations on the restriction maps:

$$r^\tau_\sigma = g_{\sigma_0} \cdot r^{\tau_{0,1}}_{\sigma_0} \cdot (g^{-1})_\tau,\ \ r^\tau_{\sigma'} = g_{\sigma_1} \cdot r^{\tau_{0,1}}_{\sigma_1} \cdot (g^{-1})_\tau. \tag{43}$$

Definition 6.1. *A diagram is a quintuple* $D = (D_0, D_1, D_{0,1}, r_0, r_1)$*, in which* (ρ_i, D_i) *is a smooth* $\tilde{E}$*-representation of* $R(\sigma_i)$*, and* $(\rho_{0,1}, D_{0,1})$ *is a smooth* $\tilde{E}$*-representation of* $R(\tau_{0,1})$*, and* $r_i \in \mathrm{Hom}_I(D_{0,1}, D_i)$*,* $i = 0, 1$.

A *morphism* between two diagrams $D = (D_0, D_1, D_{0,1}, r_0, r_1)$ and $D' = (D'_0, D'_1, D'_{0,1}, r'_0, r'_1)$ is a triple $(\psi_0, \psi_1, \eta_{0,1})$, where $\psi_i \in \mathrm{Hom}_{R(\sigma_i)}(D_i,\ D'_i)$, and $\eta_{0,1}$ lies in $\mathrm{Hom}_I(D_{0,1},\ D'_{0,1})$, and they together make the following diagrams commute as I-representations:

$$\begin{array}{ccc} D_i & \xrightarrow{\psi_i} & D'_i \\ {\scriptstyle r_i}\uparrow & & \uparrow{\scriptstyle r'_i} \\ D_{0,1} & \xrightarrow{\eta_{0,1}} & D'_{0,1} \end{array}$$

We see the set of diagrams with the morphisms defined above becomes a category, which we denote by $\mathcal{DIAG}$. The main result of this chapter can be briefly stated as:

Theorem 6.2. *The categories* $\mathcal{DIAG}$ *and* $\mathcal{COEF}_G$ *are equivalent.*

6.1.1 Homology

Let $\mathcal{V} = (V_\tau)_\tau$ be a G-equivariant coefficient system. Denote by $C_c(X_0, \mathcal{V})$ the $\tilde{E}$-vector space of all maps:

$$\omega : X_0 \to \bigcup_{\sigma \in X_0} V_\sigma,$$

such that:

- ω has finite support;
- $\omega(\sigma) \in V_\sigma$ for every vertex σ.

Denote by $C_c(X_1, \mathcal{V})$ be the $\tilde{E}$-vector space of all maps:

$$\omega : X_1 \to \bigcup_{(\sigma,\sigma') \in X_1} V_{(\sigma,\sigma')},$$

such that

- ω has finite support;
- $\omega((\sigma, \sigma')) \in V_{(\sigma,\sigma')}$.

There is an action of G on the two spaces above, induced from that of G on the tree X and $\mathcal{V}$. In more words, for an element $g \in G$,

$$g \cdot \omega(\sigma) = g_{g^{-1}\sigma}(\omega(g^{-1}\sigma)), \text{ for } \omega \in C_c(X_0, \mathcal{V});$$
$$g \cdot \omega(\tau) = g_{g^{-1}\tau}(\omega(g^{-1}\tau)), \text{ for } \omega \in C_c(X_1, \mathcal{V}).$$

The boundary map ∂ is defined as:

$$\begin{aligned} \partial: \ & C_c(X_1, \mathcal{V}) \to C_c(X_0, \mathcal{V}) \\ & \omega \mapsto (\sigma \mapsto \sum_{\tau} r_{\sigma}^{\tau}(\omega(\tau))), \end{aligned}$$

where τ goes through the edges which contain the vertex σ. It could be checked that ∂ is a G-map. Define $H_0(X, \mathcal{V})$ as the cokernel of ∂, which inherits a smooth representation of G.

6.1.2 First properties of $H_0(X, \mathcal{V})$

We fix a G-equivariant coefficient system $\mathcal{V} = (V_\tau)_\tau$ in this subsection.

Lemma 6.3. *Let ω be a 1-chain, supported on a single edge $\tau = (\sigma, \sigma')$. Then*

$$\partial(\omega) = \omega_\sigma + \omega_{\sigma'},$$

where ω_σ and $\omega_{\sigma'}$ are two 0-chains, supported respectively on σ and σ'. In more words, let $v = \omega(\tau)$, then

$$\omega_\sigma = r_\sigma^\tau(v), \text{ and } \omega_{\sigma'} = r_{\sigma'}^\tau(v).$$

Proof. This comes from the definition of boundary map ∂ directly. □

Lemma 6.4. *Let ω be a 0-chain, supported on a single vertex σ. Suppose that the two restriction maps $r_{\sigma_0}^{\tau_{0,1}}$ and $r_{\sigma_1}^{\tau_{0,1}}$ are both injective. Then the image of ω in $H_0(X, \mathcal{V})$ is non-zero.*

Proof. From the assumption and (43) above, we see every restriction map is injective. Given a non-zero 1-chain ω. If the support of ω consists of a single edge τ, then Lemma 6.3 and the injectivity of restriction maps tell that $\partial(\omega)$ is supported on the origin and terminus of τ. Otherwise, we can find at least two edges, say τ' and τ'', which are in the support of ω, and they are both the boundary of the support ω; for one endpoint σ' of τ' and another σ'' of τ'', they appear only one time as an endpoint of some edge in the support of ω. We then compute by definition the $\partial(\omega)(\sigma')$ and $\partial(\omega)(\sigma'')$. The injectivity of restriction maps guarantee they are both non-zero, i.e., $\partial(\omega)$ is supported at least at σ' and σ''. We are done.

□

Lemma 6.5. *Let ω be a 0-chain. Suppose the two restriction maps $r_{\sigma_0}^{\tau_{0,1}}$ and $r_{\sigma_1}^{\tau_{0,1}}$ are both surjective. Then, for any vertex σ, there is a 0-chain ω_σ, supported on the single vertex σ, such that,*

$$\omega + \partial C_c(X_1, \mathcal{V}) = \omega_\sigma + \partial C_c(X_1, \mathcal{V})$$

Proof. As $r_{\sigma_0}^{\tau_{0,1}}$ and $r_{\sigma_1}^{\tau_{0,1}}$ are surjective, we see every restriction map is surjective from (43). It is enough to prove the Lemma for a 0-chain which is supported on a single vertex, as every ω is a finite sum of such one.

Let $\omega_{\sigma'}$ be a 0-chain, supported on a single vertex σ', and let σ be any vertex. If $\sigma' = \sigma$, the Lemma to be proved is certainly true. Suppose we have shown the Lemma holds for the vertices whose distance from σ is smaller than m, and the path from σ' to σ is of length $m \geq 1$. Denote by τ the edge that contains the vertex σ', and which lies in the path from σ' to σ, let σ'' be the other vertex of τ. Let ω' be the 1-chain supported on τ, say $\omega'(\tau) = v$ for some $v \in V_\tau$ which satisfies $r_{\sigma'}^{\tau}(v) = \omega_{\sigma'}(\sigma')$. This is possible because $r_{\sigma'}^{\tau}$ is surjective. Let $\omega_{\sigma''}$ be the 0-chain, supported on the vertex σ'', and $\omega_{\sigma''}(\sigma'') = r_{\sigma''}^{\tau}(v)$. Then Lemma 6.3 says $\partial(\omega') = \omega_{\sigma'} + \omega_{\sigma''}$. Equivalently, $\omega_{\sigma'} + \partial C_c(X_1, \mathcal{V}) = -\omega_{\sigma''} + \partial C_c(X_1, \mathcal{V})$. Now, $-\omega_{\sigma''}$ is a 0-chain supported on the vertex σ'' which is of distance $m-1$ from σ. By induction assumption, there is 0-chain ω_σ supported on σ, such that $-\omega_{\sigma''} + \partial C_c(X_1, \mathcal{V}) = \omega_\sigma + \partial C_c(X_1, \mathcal{V})$. We are done.

□

Remark 6.6. *It is worth to note that the surjectivity of both $r_{\sigma_0}^{\tau_{0,1}}$ and $r_{\sigma_1}^{\tau_{0,1}}$ is essential here. One does not get enough information required if only one of them is surjective. In other words, there are then many 0-chains which are not necessarily congruent to a 0-chain supported on a single vertex.*

Proposition 6.7. *Suppose $r_{\sigma_0}^{\tau_{0,1}}$ and $r_{\sigma_1}^{\tau_{0,1}}$ are both isomorphisms of vector spaces.*

(1) $H_0(X, \mathcal{V}) \mid_{K_0} \cong V_{\sigma_0}$, $H_0(X, \mathcal{V}) \mid_{K_1} \cong V_{\sigma_1}$, $H_0(X, \mathcal{V}) \mid_I \cong V_{\tau_{0,1}}$

(2) *The following diagrams commute as I-representations*:

$$\begin{array}{ccc} V_{\sigma_i} & \xrightarrow{\jmath_i \cdot (ev_i)^{-1}} & H_0(X, \mathcal{V}) \\ {\scriptstyle r_{\sigma_i}^{\tau_{0,1}}} \uparrow & & \uparrow {\scriptstyle Id} \\ V_{\tau_{0,1}} & \xrightarrow{\iota} & H_0(X, \mathcal{V}) \end{array}$$

where $i = 0, 1$.

Proof. For $i = 0, 1$, denote by $C_c(\sigma_i, \mathcal{V})$ the vector space of 0-chains which are supported on the single vertex σ_i. We then have an evaluation map ev_i, which is an isomorphism of $R(\sigma_i)$-representations:

$$\begin{aligned} ev_i: \ & C_c(\sigma_i, \mathcal{V}) \to V_{\sigma_i} \\ & \omega \mapsto (-1)^{\delta(\sigma_i)-1} \omega(\sigma_i), \end{aligned}$$

where $\delta(\sigma_i)$ is the period of σ_i.

Let $\jmath_i$ be the composition of the inclusion $C_c(\sigma_i, \mathcal{V}) \to C_c(X_0, \mathcal{V})$ and the canonical map $C_c(X_0, \mathcal{V}) \to H_0(X, \mathcal{V})$. It's certainly an $R(\sigma_i)$-map. Moreover, Lemma 6.4 and Lemma 6.5 imply that $\jmath_i$ is indeed an isomorphism of vector spaces. We get the isomorphism $\jmath_i \cdot (ev_i)^{-1} : V_{\sigma_i} \to H_0(X, \mathcal{V}) \mid_{R(\sigma_i)}$. As $r_{\sigma_i}^{\tau_{0,1}}$ are isomorphisms of I-representations, and $I \subset K_i$, we see $\iota = \jmath_i \cdot (ev_i)^{-1} \cdot r_{\sigma_i}^{\tau_{0,1}} : V_{\tau_{0,1}} \to H_0(X, \mathcal{V}) \mid_I$ is an isomorphism of I-representations. We have shown (1)

(2) follows from the construction of (1).

□

6.1.3 Constant functor

Let Rep_G be the category of smooth $\tilde{E}$-representations of G. Let π be a smooth representation of G, with underlying space W. Let σ be a simplex on the tree X, and we set

$$(\mathcal{K}_\pi)_\sigma = W.$$

If $\sigma \subseteq \sigma'$ are two simplices, the restriction map $r_\sigma^{\sigma'}$ is defined as Id_W. For every $g \in G$, and every simplex σ in X, the linear map g_σ is defined by:

$$\begin{aligned} g_\sigma : (\mathcal{K}_\pi)_\sigma &\to (\mathcal{K}_\pi)_{g\sigma} \\ v &\mapsto \pi(g) \cdot v. \end{aligned}$$

This G-equivariant coefficient system defined on X is denoted by $\mathcal{K}_\pi$.

Lemma 6.8. *Let π be a smooth representation of G. Then*

$$H_0(X, \mathcal{K}_\pi) \cong \pi$$

as G-representations.

Proof. Define an evaluation map ev from $C_c(X_0, \mathcal{K}_\pi)$ to π:

$$\begin{aligned} ev : \ & C_c(X_0, \mathcal{K}_\pi) \to \pi \\ & \omega \mapsto \sum_{\sigma \in X_0} (-1)^{\delta(\sigma)-1} \cdot \omega(\sigma), \end{aligned}$$

where $\delta(\sigma)$ is the period of σ. ev is well-defined as every ω is of finite support. It's easy to check that ev is a G-map.

As the restriction maps are Id_W, we see from Lemma 6.3 that, ev is trivial on the image of the boundary map ∂. Hence ev induces a G-map:

$$ev : \ H_0(X, \mathcal{K}_\pi) \to \pi.$$

We need to show the above G-map is also an isomorphism of vector spaces. We note that $(\mathcal{K}_\pi)_{\sigma_i} = W$, then we see

$$ev \mid C_c(\sigma_i, \mathcal{K}_\pi) = ev_i$$

by our definitions, i.e., $ev_i = ev \circ \jmath_i$, $i = 0, 1$. Here ev_i and $\jmath_i$ are the maps defined in the last subsection (for $\mathcal{V} = \mathcal{K}_\pi$). As the restriction maps are Id_W, we have observed that $\jmath_i$ is an isomorphism of vector space in the argument of Proposition 6.7, which gives us $ev = ev_i \circ \jmath_i^{-1} : H_0(X, \mathcal{K}_\pi) \to (\mathcal{K}_\pi)_{\sigma_i}$ is as desired. $\square$

Proposition 6.9. *Let $\mathcal{V} = (V_\sigma)_\sigma$ be a G-equivariant coefficient system. Let (π, W) be a smooth representation of G. Then*

$$\mathrm{Hom}_{\mathcal{COEF}_G}(\mathcal{V}, \mathcal{K}_\pi) \cong \mathrm{Hom}_G(H_0(X, \mathcal{V}), \pi)$$

Proof. By Lemma 6.8, $H_0(X, \mathcal{K}_\pi) \cong \pi$. Any morphism between G-equivariant coefficient systems induces a homomorphism between the corresponding 0-homology which is compatible with the action of G, i.e., there is a map:

$$\mathrm{Hom}_{\mathcal{COEF}_G}(\mathcal{V}, \mathcal{K}_\pi) \to \mathrm{Hom}_G(H_0(X, \mathcal{V}), \pi),$$

and we need to construct an inverse of this map.

Let $\phi \in \mathrm{Hom}_G(H_0(X, \mathcal{V}), \pi)$. Given a vertex σ and a vector v in V_σ, let $\omega_{\sigma,\, v}$ be the 0-chain, such that

$$\mathrm{Supp}\ \omega_{\sigma,\, v} \subseteq \sigma,\ \omega_{\sigma,\, v}(\sigma) = v.$$

For the simplex σ, we then define

$$\begin{aligned} \phi_\sigma :\ V_\sigma &\to W \\ v &\mapsto \phi(\omega_{\sigma,\, v} + \partial C_c(X_1, \mathcal{V})). \end{aligned}$$

For an edge τ on the tree X, with endpoints σ and σ', we define

$$\begin{aligned} \phi_\tau :\ V_\tau &\to W \\ v' &\mapsto (-1)^{\delta(\sigma)-1} \phi_\sigma(r^\tau_\sigma(v')). \end{aligned}$$

The independence of the choice of the vertex σ in the definition of ϕ_τ comes from Lemma 6.3, i.e.,

$$\phi(\omega_{\sigma,\, r^\tau_\sigma(v')} + \partial C_c(X_1, \mathcal{V})) = \phi(-\omega_{\sigma',\, r^\tau_{\sigma'}(v')} + \partial C_c(X_1, \mathcal{V})).$$

Then this variety of linear maps $(\phi_\sigma)_\sigma$ consists of a morphism from the coefficient system $\mathcal{V}$ to $\mathcal{K}_\pi$, furthermore it respects the actions of G on them. One can check that it induces ϕ on the 0-homology without difficulty. $\square$

There is a natural functor from the category of coefficient systems to that of diagrams:

Definition 6.10. *Let $\mathcal{D}$ be the functor from $\mathcal{COEF}_G$ to $\mathcal{DIAG}$:*

$$\begin{gathered}\mathcal{D}:\ \mathcal{COEF}_G \to \mathcal{DIAG}\\ \mathcal{V} = (V_\sigma)_\sigma \mapsto (V_{\sigma_0}, V_{\sigma_1}, V_{\tau_{0,1}}, r_{\sigma_0}^{\tau_{0,1}}, r_{\sigma_1}^{\tau_{0,1}})\end{gathered}$$

We will construct an inverse $\mathcal{C}$ of $\mathcal{D}$ in the following subsections. Fix an object $D = (D_0, D_1, D_{0,1}, r_0, r_1)$ in $\mathcal{DIAG}$.

6.1.4 Underlying vector space

From the diagram D above, we can form the following compactly induced representations:

$$\mathrm{ind}_{K_0}^G \rho_0,\ \mathrm{ind}_{K_1}^G \rho_1,\ \mathrm{ind}_I^G \rho_{0,1}.$$

For a vertex $\sigma \in X_0$, with period $\delta(\sigma)$, there is $g \in G$, such that $\sigma = g\sigma_{\delta(\sigma)-1}$. We then define

$$F_\sigma = \{f \in \mathrm{ind}_{K_{\delta(\sigma)-1}}^G \rho_{\delta(\sigma)-1} :\ \mathrm{Supp}\, f \subseteq K_{\delta(\sigma)-1} g^{-1}\}$$

For an edge $\tau \in X_1$, there is a $g \in G$, such that $\tau = g\tau_{0,1}$. We define

$$F_\tau = \{f \in \mathrm{ind}_I^G \rho_{0,1} :\ \mathrm{Supp}\, f \subseteq I g^{-1}\}$$

6.1.5 Restriction maps

To define the restriction maps, we start with two fundamental ones, $r_{\sigma_i}^{\tau_{0,1}}$, $i = 0, 1$, and then we extend them to the general case by translations.

For $i = 0, 1$, the evaluation map ev_i from F_{σ_i} to D_i is naturally an isomorphism of K_i-representations, defined by

$$\begin{gathered}ev_i :\ F_{\sigma_i} \to D_i\\ f \mapsto f(1),\end{gathered}$$

whose inverse ev_i^{-1} is

$$\begin{gathered}ev_i^{-1} :\ D_i \to F_{\sigma_i}\\ v \mapsto f_v,\end{gathered}$$

where f_v is supported on R_{σ_i}, and $f_v(k) = \rho_i(k)v$, for $k \in R_{\sigma_i}$.

Similarly, we have isomorphisms $ev_{0,1}$ and $ev_{0,1}^{-1}$ of I-representations,

$$\begin{aligned} ev_{0,1} :\ F_{\tau_{0,1}} &\to D_{0,1} \\ f &\mapsto f(1), \end{aligned}$$

and

$$\begin{aligned} ev_{0,1}^{-1} :\ D_{0,1} &\to F_{\tau_{0,1}} \\ v &\mapsto f_v, \end{aligned}$$

where f_v is supported on I, and $f_v(i) = \rho_{0,1}(i)v$ for $i \in I$.

Let $r_{\sigma_i}^{\tau_{0,1}} = ev_i^{-1} \circ r_i \circ ev_{0,1}$, for $i = 0, 1$. It is an I-map from $F_{\tau_{0,1}}$ to F_{σ_i}. For later application, we write down $r_{\sigma_i}^{\tau_{0,1}}$ explicitly: on f_v, for any $v \in D_{0,1}$, we have

$$r_{\sigma_i}^{\tau_{0,1}}(f_v) = f_{r_i(v)}. \tag{44}$$

In summary, we get a diagram $\tilde{D} = (F_{\sigma_0}, F_{\sigma_1}, F_{\tau_{0,1}}, r_{\sigma_0}^{\tau_{0,1}}, r_{\sigma_1}^{\tau_{0,1}})$, and D is isomorphic to $\tilde{D}$ via the morphism $ev = (ev_0, ev_1, ev_{0,1})$.

Let τ be an edge, containing a vertex σ. Then there exists $g \in G$ such that

$$\tau = g\tau_{0,1},\ \sigma = g\sigma_{\delta(\sigma)-1}, \tag{45}$$

in which we note that the choice of g is up to right multiplication by an element of I.

Define the restriction map r_σ^τ from F_τ to F_σ as:

$$\begin{aligned} r_\sigma^\tau :\ F_\tau &\to F_\sigma \\ f &\mapsto g \cdot r_{\sigma_{\delta(\sigma)-1}}^{\tau_{0,1}}(g^{-1} \cdot f). \end{aligned}$$

We need to verify the definition above is independent of the choice of g. But this is immediate: any other choice g' differs from g by an element $i \in I$, and the result comes from the fact that $r_{\sigma_{\delta(\sigma)-1}}^{\tau_{0,1}}$ is an I-map. We conclude from (44) that $r_\sigma^\tau(f) = g \cdot f_{r_{\delta(\sigma)-1}(v)}$, where $v = f(g^{-1})$.

For any simplex τ, let $r_\tau^\tau = \mathrm{Id}_{F_\tau}$.

6.1.6 G-action

In 6.1.4, for any simplex τ, $g \cdot f$ has been defined, for any $g \in G$ and $f \in F_\tau$, from which there is a linear map:

$$\begin{aligned} g_\tau :\ F_\tau &\to F_{g\tau} \\ f &\mapsto gf \end{aligned}$$

Certainly, 1_τ is the identity map and $g_{h\tau} \circ h_\tau = (gh)_\tau$. We still need to check the linear maps above are compatible with the restriction maps in 6.1.5. In other words, for an edge τ, containing a vertex σ, the following diagram is commutative

$$\begin{array}{ccc} F_\sigma & \xrightarrow{g_\sigma} & F_{g\sigma} \\ {\scriptstyle r_\sigma^\tau}\uparrow & & \uparrow{\scriptstyle r_{g\sigma}^{g\tau}} \\ F_\tau & \xrightarrow{g_\tau} & F_{g\tau} \end{array}$$

for any $g \in G$.

On the one hand, from (44), for a chosen g' satisfying the condition in (45), $g_\tau \cdot r_\sigma^\tau(f) = gg' \cdot f_{r_{\delta(\sigma)-1}(v)}$, where $v = f(g'^{-1})$. On another hand, $r_{g\sigma}^{g\tau} \cdot g_\tau(f) = r_{g\sigma}^{g\tau}(g \cdot f) = gg' f_{r_{\delta(\sigma)-1}(v')}$, where $v' = g \cdot f((gg')^{-1}) = v$. We are done.

In summary, we have associated a G-equivariant system $\mathcal{F} = (F_\sigma)_\sigma$ to a diagram $\mathcal{D}$.

6.1.7 Morphisms

Let $D = (D_0, D_1, D_{0,1}, r_0, r_1)$ and $D' = (D'_0, D'_1, D'_{0,1}, r'_0, r'_1)$ be two diagrams, and $\psi = (\psi_0, \psi_1, \eta_{0,1})$ be a morphism between them. Let $\mathcal{F} = (F_\sigma)_\sigma$ and $\mathcal{F}' = (F'_\sigma)_\sigma$ be the coefficient systems associated to D and D'.

Let σ be a vertex, and let $g \in G$ be such that $\sigma = g\sigma_{\delta(\sigma)-1}$. For $f \in F_\sigma$, let $v = f(g^{-1})$, then we define

$$\begin{aligned} \psi_\sigma :\ & F_\sigma \to F'_\sigma \\ & f \mapsto g \cdot f_{\psi_{\delta(\sigma)-1}(v)}, \end{aligned}$$

where $f_{\psi_{\delta(\sigma)-1}(v)}$ is the unique function in F'_σ such that $f_{\psi_{\delta(\sigma)-1}(v)}(1) = \psi_{\delta(\sigma)-1}(v)$.

Let τ be an edge, and let $g \in G$ be such that $\tau = g\tau_{0,1}$. Similarly, for $f \in F_\tau$, let $v = f(g^{-1})$, and we define:

$$\begin{aligned} \psi_\tau :\ & F_\tau \to F'_\tau \\ & f \mapsto g \cdot f_{\eta_{0,1}(v)}, \end{aligned}$$

where $f_{\eta_{0,1}(v)}$ is the unique function in F'_τ such that $f_{\eta_{0,1}(v)}(1) = \eta_{0,1}(v)$. It is immediate to check the definition does not depend on the choice of g.

In summary, we have a collection of linear maps $(\psi_\tau)_\tau$. We need to verify they are compatible with the restriction maps and the G-actions, i.e., the following two diagrams commute: in the first one, τ is an edge containing a vertex σ, and in the second, τ is any simplex, $h \in G$.

$$\begin{array}{ccc} F_\sigma & \xrightarrow{\psi_\sigma} & F'_\sigma \\ {\scriptstyle r_\sigma^\tau}\uparrow & & \uparrow{\scriptstyle (r')_\sigma^\tau} \\ F_\tau & \xrightarrow{\psi_\tau} & F'_\tau \end{array} \qquad \begin{array}{ccc} F_{h\tau} & \xrightarrow{\psi_{h\tau}} & F'_{h\tau} \\ {\scriptstyle h_\tau}\uparrow & & \uparrow{\scriptstyle h\tau} \\ F_\tau & \xrightarrow{\psi_\tau} & F'_\tau \end{array}$$

We begin with the first. Given $f \in F_\tau$, $\psi_\sigma \circ r^\tau_\sigma(f) = \psi_\sigma(g \cdot f_{r_{\delta(\sigma)-1}(v)})$, where $v = f(g^{-1})$, for a chosen g satisfying (45). As $g \cdot f_{r_{\delta(\sigma)-1}(v)}(g^{-1}) = r_{\delta(\sigma)-1}(v)$, we get $\psi_\sigma \circ r^\tau_\sigma(f) = g \cdot f_{\psi_{\delta(\sigma)-1} \cdot r_{\delta(\sigma)-1}(v)}$. On the other hand, $(r')^\tau_\sigma \circ \psi_\tau(f) = (r')^\tau_\sigma(g \cdot f_{\eta_{0,1}(v)})$. As $g \cdot f_{\eta_{0,1}(v)}(g^{-1}) = \eta_{0,1}(v)$, we see $(r')^\tau_\sigma(g \cdot f_{\eta_{0,1}(v)}) = g \cdot f_{r'_{\delta(\sigma)-1}(\eta_{0,1}(v))}$. We note that $v \in D_{0,1}$. It's then certainly $\psi_{\delta(\sigma)-1} \cdot r_{\delta(\sigma)-1}(v) = r'_{\delta(\sigma)-1}(\eta_{0,1}(v))$, as ψ is a morphism of diagrams.

For the second diagram, given $f \in F_\tau$, we note that $h \cdot f((hg)^{-1}) = f(g^{-1}) = v$, then its commutativity comes directly from definitions.

We have constructed a functor $\mathcal{C}$ from the category of diagrams to that of G-equivariant systems. We write as a definition:

Definition 6.11. *Let $\mathcal{C}$ be the functor:*

$$\mathcal{C}: \mathcal{DIAG} \to \mathcal{COEF}_G$$
$$D \mapsto \text{the coefficient system } \mathcal{C}(D) = (F_\tau)_\tau \text{ constructed above.}$$

6.1.8 The equivalence of Diagrams with Coefficient systems

We begin to prove Theorem 6.2, in which the equivalence is induced from the functors $\mathcal{D}$ and $\mathcal{C}$.

We verify first the functor $\mathcal{C}$ preserves the composition of morphisms of objects.

Let $\psi : D \to D'$ and $\psi' : D' \to D''$ be two morphisms of diagrams. We have to check $\mathcal{C}(\psi' \circ \psi) = \mathcal{C}(\psi') \circ \mathcal{C}(\psi)$.

For a vertex σ, let $g \in G$ be such that $\sigma = g\sigma_{\delta(\sigma)-1}$. For $f \in F_\sigma$, let $v = f(g^{-1})$. Then we have $(\psi' \circ \psi)_\sigma(f) = g \cdot f_{(\psi' \circ \psi)_{\delta(\sigma)-1}(v)} = g \cdot f_{\psi'_{\delta(\sigma)-1} \cdot (\psi_{\delta(\sigma)-1})(v)}$. As $g \cdot f_{\psi_{\delta(\sigma)-1}(v)}(g^{-1}) = \psi_{\delta(\sigma)-1}(v)$, we see $\psi'_\sigma \cdot (\psi_\sigma(f)) = \psi'_\sigma(g \cdot f_{\psi_{\delta(\sigma)-1}(v)}) = g \cdot f_{\psi'_{\delta(\sigma)-1} \cdot (\psi_{\delta(\sigma)-1})(v)}$. Hence $(\psi' \circ \psi)_\sigma = \psi'_\sigma \cdot \psi_\sigma$.

For an edge τ, $(\psi' \circ \psi)_\tau = \psi'_\tau \cdot \psi_\tau$ holds similarly.

It remains for us to verify that $\mathcal{D} \circ \mathcal{C}$ (resp, $\mathcal{C} \circ \mathcal{D}$) is isomorphic to $\mathrm{Id}_{\mathcal{DIAG}}$ (resp, $\mathrm{Id}_{\mathcal{COEF}_G}$) as functors of categories.

For a diagram $D = (D_0, D_1, D_{0,1}, r_0, r_1)$, from the definition of the functors $\mathcal{D}$ and $\mathcal{C}$, we see

$$\mathcal{D} \circ \mathcal{C}(D) = \tilde{D} = (F_{\sigma_0}, F_{\sigma_1}, F_{\tau_{0,1}}, r^{\tau_{0,1}}_{\sigma_0}, r^{\tau_{0,1}}_{\sigma_1}),$$

and we have already seen that $ev = (ev_0, ev_1, ev_{0,1})$ is an isomorphism from $\tilde{D}$ to D.

We now verify that ev induces an isomorphism from the functor $\mathcal{D} \circ \mathcal{C}$ to $\mathrm{Id}_{\mathcal{DIAG}}$.

Let $D' = (D'_0, D'_1, D'_{0,1}, r'_0, r'_1)$ be another diagram, and let $\psi = (\psi_0, \psi_1, \eta_{0,1})$ be a morphism from D to D'. Let $\tilde{D}' = \mathcal{D} \circ \mathcal{C}(D') =$

$$(F'_{\sigma_0}, F'_{\sigma_1}, F'_{\tau_{0,1}}, (r')^{\tau_{0,1}}_{\sigma_0}, (r')^{\tau_{0,1}}_{\sigma_1})$$

We are reduced to check the following diagrams are commutative: for $i = 0, 1$

$$\begin{tikzcd} F_{\sigma_i} \arrow[r, "\mathcal{D}\circ\mathcal{C}(\psi_i)"] \arrow[d, "ev_i"'] & F'_{\sigma_i} \arrow[d, "ev'_i"] \\ D_i \arrow[r, "\psi_i"] & D'_i \end{tikzcd} \qquad \begin{tikzcd} F_{\tau_{0,1}} \arrow[r, "\mathcal{D}\circ\mathcal{C}(\eta_{0,1})"] \arrow[d, "ev_{0,1}"'] & F'_{\tau_{0,1}} \arrow[d, "ev'_{0,1}"] \\ D_{0,1} \arrow[r, "\eta_{0,1}"] & D'_{0,1} \end{tikzcd}$$

We do the first as an example. Given $f \in F_{\sigma_i}$, we see $\psi_i \cdot ev_i(f) = \psi_i(f(1))$. By our definition of $\mathcal{C}(\psi)$ in 6.1.7 and that of $\mathcal{D}$, we know $\mathcal{D} \circ \mathcal{C}(\psi)(f) = f_{\psi_i(f(1))}$, therefore $ev'_i(\mathcal{D} \circ \mathcal{C}(\psi)(f)) = \psi_i(f(1))$. The second follows similarly.

Let $\mathcal{V} = (V_\tau)_\tau$ be a G-equivariant system, with restriction maps t^τ_σ. Let $\mathcal{F} = (F_\tau)_\tau$ be the coefficient system $\mathcal{C} \circ \mathcal{D}(\mathcal{V})$, with restriction maps r^τ_σ. We are going to construct an isomorphism $ev = (ev_\tau)_\tau$ from $\mathcal{F}$ to $\mathcal{V}$, which induces an isomorphism of functors from $\mathcal{C} \circ \mathcal{D}$ to $\mathrm{Id}_{\mathcal{COEF}_G}$.

Let τ be an edge containing a vertex σ. There exists $g \in G$ such that $\tau = g\tau_{0,1}$ and $\sigma = g\sigma_{\delta(\sigma)-1}$.

For the vertex σ, define ev_σ:

$$\begin{aligned} ev_\sigma :\ & F_\sigma \to V_\sigma \\ & f \mapsto g_{\sigma_{\delta(\sigma)-1}} \cdot v, \end{aligned}$$

where $v = f(g^{-1}) \in V_{\sigma_{\delta(\sigma)-1}}$.

For the edge τ, define ev_τ:

$$\begin{aligned} ev_\tau :\ & F_\tau \to V_\tau \\ & f \mapsto g_{\tau_{0,1}} \cdot v, \end{aligned}$$

where $v = f(g^{-1}) \in V_{\tau_{0,1}}$. Certainly, ev_σ and ev_τ are both linear maps of vector spaces. However, they are indeed isomorphisms of vector spaces, as one notes that $ev_{\sigma_i} = ev_i$, $ev_{\tau_{0,1}} = ev_{0,1}$ are isomorphisms.

We need to verify the definition above is independent of the choice of g. Let $g' = g \cdot i$ for some $i \in I$, $v' = f(g'^{-1})$. Then $g'_{\sigma_{\delta(\sigma)-1}} \cdot v' = g_{\sigma_{\delta(\sigma)-1}} \cdot i_{\sigma_{\delta(\sigma)-1}} v'$. But, as $f \in F_\sigma$, we see $i_{\sigma_{\delta(\sigma)-1}} f(i^{-1}g^{-1}) = f(g^{-1}) = v$. Similarly, ev_τ is also independent of the choice of g.

We turn to show $(ev_\tau)_\tau$ is compatible with the G-actions. For an element $g' \in G$ and $f \in F_\sigma$, $ev_{g'\sigma} \cdot g'_\sigma(f) = ev_{g'\sigma}(g' \cdot f) = (g'g)_{\sigma_{\delta(\sigma)-1}} \cdot v$, where $v = f(g^{-1})$. On the other hand, $g'_\sigma \cdot ev_\sigma(f) = g'_\sigma(g_{\sigma_{\delta(\sigma)-1}} v)$. But

$(g'g)_{\sigma_{\delta(\sigma)-1}} = g'_\sigma \cdot g_{\sigma_{\delta(\sigma)-1}}$, as $\sigma = g \cdot \sigma_{\delta(\sigma)-1}$. For an edge τ, $ev_{g'\tau} \cdot g'_\tau = g'_\tau \cdot ev_\tau$ follows in the same way.

It remains for us to show $(ev_\tau)_\tau$ are compatible with the restriction maps, i.e., in the same notations above, to check the following diagram is commutative:

$$\begin{tikzcd} F_\tau \arrow[r, "ev_\tau"] \arrow[d, "r^\tau_\sigma"'] & V_\tau \arrow[d, "t^\tau_\sigma"] \\ F_\sigma \arrow[r, "ev_\sigma"] & V_\sigma \end{tikzcd}$$

Given $f \in F_\tau$, let $v = f(g^{-1})$. Hence $t^\tau_\sigma \cdot ev_\tau(f) = t^\tau_\sigma \cdot (g_{\tau_{0,1}} \cdot v)$. On the other hand, as $\mathcal{F}$ comes from the diagram $(V_{\sigma_0}, V_{\sigma_1}, V_{\tau_{0,1}}, t^{\tau_{0,1}}_{\sigma_0}, t^{\tau_{0,1}}_{\sigma_1})$, we see $ev_\sigma \cdot r^\tau_\sigma(f) = ev_\sigma(g \cdot f_{t^{\tau_{0,1}}_{\sigma_{\delta(\sigma)-1}}(v)}) = g_{\sigma_{\delta(\sigma)-1}} \cdot t^{\tau_{0,1}}_{\sigma_{\delta(\sigma)-1}}(v)$, where we have used the remark at the end of 6.1.5 and the definition of ev_σ. But $t^\tau_\sigma \cdot g_{\tau_{0,1}} = g_{\sigma_{\delta(\sigma)-1}} \cdot t^{\tau_{0,1}}_{\sigma_{\delta(\sigma)-1}}$ is certainly true, as the G-actions and the restriction maps are compatible on a coefficient system by definition.

We arrive at the final step, i.e., to show ev induces an isomorphism of functors from $\mathcal{C} \circ \mathcal{D}$ to $\mathrm{Id}_{\mathcal{COEF}_G}$.

Let $\mathcal{V}' = (V'_\tau)_\tau$ be another G-equivariant system. Let $(\phi_\tau)_\tau$ be a morphism from $\mathcal{V}$ to $\mathcal{V}'$. Therefore, we get a new coefficient system $\mathcal{F}' = \mathcal{C} \circ \mathcal{D}(\mathcal{V}') = (F'_\tau)_\tau$, with restriction maps r^τ_σ. We are reduced to check the following two diagrams are commutative:

$$\begin{tikzcd} F_\sigma \arrow[r, "(\mathcal{D}(\phi))_\sigma"] \arrow[d, "ev_\sigma"'] & F'_\sigma \arrow[d, "ev'_\sigma"] \\ V_\sigma \arrow[r, "\phi_\sigma"] & V'_\sigma \end{tikzcd} \qquad \begin{tikzcd} F_\tau \arrow[r, "(\mathcal{D}(\phi))_\tau"] \arrow[d, "ev_\tau"'] & F'_\tau \arrow[d, "ev'_\tau"] \\ V_\tau \arrow[r, "\phi_\tau"] & V'_\tau \end{tikzcd}$$

Here $\mathcal{D}(\phi)$ is the morphism of diagrams from $\mathcal{D}(\mathcal{V})$ to $\mathcal{D}(\mathcal{V}')$, induced from ϕ. There is no essential difference with that we have just done in the converse direction, so we don't show details again.

We have proved Theorem 6.2.

7 Appendix B: Some p-adic principal series

In the appendix, with an eye on the mysterious p-adic Banach space representation theory, we collect some observations obtained so far in the course of this thesis. In the first section, using a result of Ardakov, we verify the Iwasawa algebra of N_1 has the same global dimension and Krull dimension, when F is an unramified extension of $\mathbf{Q}_p$. In the second section, we prove a sufficient condition for the irreducibility of p-adic principal series of $U(1,1)(\mathbf{Q}_{p^2}/\mathbf{Q}_p)$, by modifying a method of Schneider and Teitelbaum.

7.1 The Iwasawa algebra of N_1

In this section we assume F is the unramified extension of $\mathbf{Q}_p$ of degree d. We define another filtration $\{M_k\}_{k\in\mathbb{Z}}$ on the upper unipotent subgroup N, say,

$$M_k = \{n(x,y):\ x,y \in \mathfrak{p}_E^k\}$$

One notes that $M_1 = N_1$ in our previous notation, and M_k is an open normal subgroup of M_0, when $k \geq 1$. In all, they together form a filtration of open normal neighbourhoods of the identity of M_0, but M_1 has clearly many more open normal subgroups.

All the unexplained terminologies appeared in this section can be found in the book [DdSMS99], or in the survey paper [AB06]. We also give more precise references in the following.

Proposition 7.1. *M_1 is a uniform pro-p group of dimension $3d$.*

Proof. For the purpose of later calculation, we record the following lemma whose proof is a simple calculation.

Lemma 7.2. (1). $[n(x,y),\ n(x_1,y_1)] = n(0, \bar{x}_1 x - \bar{x} x_1)$
(2). $n(x,y)^k = n(kx, ky - \frac{k(k-1)}{2} x\bar{x})$*, for* $k \geq 0$.

Based on the above Lemma, we can check the lower p-series of M_1 is exactly the filtration $\{M_k\}_{k\geq 1}$ and M_1 is powerful ($p \neq 2$!), noting that any element in M_{k+1} could lift uniquely to a p-th root in M_k. Using the decomposition that

$$n(x,y) = n(x, -\tfrac{x\bar{x}}{2}) \cdot n(0, y + \tfrac{x\bar{x}}{2})$$

we see immediately that M_1 is topologically finitely generated (see Lemma 7.5 below). The following observation completes the proof of our proposition.

Lemma 7.3. *The index of M_{k+1} in M_k is q^3, for $k \geq 1$.*

Proof. This is just simple counting. □

□

Remark 7.4. *The group M_0 is not powerful and M_1 is indeed nilpotent but non-commutative. When $d = 1$, M_1 is also Heisenberg.*

Lemma 7.5. *A minimal set S of generators (topologically) for M_1 is given by*

$$\{c_i = n(0, p(\eta - \bar{\eta})\eta_1^i),\ d_i = n(p\eta_1^i, -\tfrac{p^2\eta_1^{2i}}{2}),\ d_i' = n(p\eta\eta_1^i, -\tfrac{p^2\eta_1^{2i+1}}{2})\}_{0\leq i\leq d-1},$$

where η is a root of unity of order q^2-1 in E and $\eta_1 = N_{E/F}(\eta)$.

Proof. This is from $n(x,y) = n(x, -\frac{\bar{x}x}{2}) \cdot n(0, y + \frac{\bar{x}x}{2})$ and the structure of local fields. $\square$

Proposition 7.6. *The $\mathbf{Q}_p$-Lie algebra $\mathcal{L}(M_1)$ [12] of M_1 has length at least $3d$.*

Proof. We construct a filtration of sub-Lie algebras of length $3d$ for the $\mathbf{Z}_p$-Lie algebra $L(M_1)$ ([DdSMS99], 4.29), which is provided by last Lemma. We start with description of operations of $L(M_1)$, which involves some calculations.

$$\begin{aligned} \text{Addition} : n(x,y) + n(x_1,y_1) &= n(x+x_1, y+y_1 - \tfrac{1}{2}(x\bar{x}_1 + \bar{x}x_1)) \\ \text{Lie bracket} : [n(x,y), n(x_1,y_1)] &= n(0, x_1\bar{x} - \bar{x}_1 x) \end{aligned}$$

Let S be the set given in Lemma 7.5 and S' be a subset of S which contains $C(S) = \{n(0, p(\eta - \bar{\eta})\eta_1^i)\}_{0 \le i \le d-1}$. We then claim the sub-Lie algebra of $L(M_1)$ generated by S' is just the $\mathbf{Z}_p$-submodule of $L(M_1)$ generated by S'. Firstly the square of any element in S is zero, which is directly from the definition of the Lie bracket. Secondly, $C(S)$ lies in the center of $L(M_1)$. Now the product of any two elements in $S \backslash C(S)$ is in the submodule generated by $C(S)$. The claim is done.

Let $S_k = \{c_i : 1 \le i \le k\}$ for $1 \le k \le d$. Next, for $d+1 \le k \le 2d$, let $S_k = S_d \cup \{d_i : 1 \le i \le k-d\}$. For $2d+1 \le k \le 3d$, let $S_k = S_{2d} \cup \{d'_i : 1 \le i \le k-2d\}$. Clearly $S_k \subset S_{k'}$ when $k \le k'$ and we denote by $\langle S'' \rangle$ the sub-Lie algebra generated by S''. Then the claim above guarantees the filtration $0 \subset \langle S_1 \rangle \subset \ldots \subset \langle S_{3d} \rangle = L(M_1)$ is of length $3d$. We are done. $\square$

From the argument of last proposition we see $\mathcal{L}(M_1)$ is indeed soluble.

Corollary 7.7. *The completed group ring $\mathbf{F}_p[[M_1]]$ has the same Krull dimension and global dimension, which is just the dimension $3d$ of M_1 as a compact p-adic Lie group.*

Proof. This is from Theorem A in [Ard04], with the last Proposition. $\square$

7.2 Irreducibility of p-adic principal series of $U(1,1)(\mathbf{Q}_{p^2}/\mathbf{Q}_p)$

In this section, we investigate briefly some example in which we form a sufficient condition for the irreducibility (topologically) of principal series for the group $G = U(1,1)(\mathbf{Q}_{p^2}/\mathbf{Q}_p)$. It satisfies the Iwasawa decomposition, say $G = BK$, where B is the subgroup of upper triangular matrices, and K is

[12] [DdSMS99], 9.5

the hyperspecial maximal compact subgroup of G (unique up to conjugacy). Then we are reduced to looking at the principal series representations of K. Actually, what we have really done is to prove a sufficient condition for simplicity of induced modules M_χ (defined below) of $L[[K]]$, where we follow [ST02] closely. Then, by duality of Schneider and Teitelbaum ([ST02], Corollary 3.6), we obtain irreducibility result for principal series of K.

We start by recalling some notations. Let I be the standard Iwahori subgroup of G, and let N_1' be the lower unipotent subgroup of I, i.e., it consists of matrices of the form:

$$n'(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

where, $x = -\bar{x} \in \mathfrak{p}_{\mathbf{Q}_{p^2}}$. Hence, N_1' is a pro-cyclic group generated by a single element $\gamma = n'(p(\eta - \bar{\eta}))$, where η is a root of unity of order $p^2 - 1$ in $\mathbf{Q}_{p^2}$. Fix a finite extension L of $\mathbf{Q}_p$ which contains $\mathbf{Q}_{p^2}$. Then the Iwasawa algebra $L[[N_1']] = L \otimes_{\mathfrak{o}_L} \mathfrak{o}_L[[N_1']]$ is isomorphic to the ring of formal power series in $\gamma - 1$ with bounded coefficients in L.

Denote by H_0 and N_0 respectively the diagonal and upper unipotent subgroups of I. The upper triangular subgroup B_0 of I is the semi-direct product of H_0 and N_0. Let χ be a continuous character of $U_{\mathbf{Q}_{p^2}}$, taking values in $L^\times$ (hence indeed in $\mathfrak{o}_L^\times$). Up to a symbol, χ is determined by $c_1 = \chi(1+p)$ and $c_2 = \chi(1+\eta p)$. The restriction of χ to $1 + p\mathbf{Z}_p$ is then determined by c_1, and there exists a constant $c(\chi) \in L$ such that

$$\chi(1+x) = (1+x)^{c(\chi)},$$

for a small enough integer $x \in \mathbf{Z}_p$.

The character χ extends uniquely to a continuous homomorphism of L-algebras $\chi : L[[H_0]] \to L$. The projection from B_0 to H_0 induces a continuous algebra epimorphism from $L[[B_0]]$ to $L[[H_0]]$. Denote by $L_{B_0,\chi}$ the composite homomorphism from $L[[B_0]]$ to L. Form the induced modules N_χ of $L[[I]]$ and M_χ of $L[[K]]$:

$$N_\chi = L[[I]] \otimes_{L[[B_0]]} L_{B_0,\chi}, \quad M_\chi = L[[K]] \otimes_{L[[B_0]]} L_{B_0,\chi}.$$

The product homeomorphism $N_1' \times B_0 \to I$ gives rise to an isomorphism of Iwasawa algebras, where the right-hand side is the completed tensor product of linear topologically $\mathfrak{o}$-modules ([SGA70], $\mathrm{VII}_B(0.3)$):

$$\mathfrak{o}_L[[I]] \cong \mathfrak{o}_L[[N_1']] \widehat{\otimes} \mathfrak{o}_L[[B_0]].$$

Therefore, $N_\chi \cong L[[N_1']]$ as an $L[[N_1']]$-module. By the inclusion $L[[N_1']] \subseteq L[[I]]$, any $L[[I]]$-submodule of N_χ then corresponds to some ideal of $L[[N_1']]$.

We note that $L[[N_1']]$ is a PID as N_1' is generated by γ, and every ideal is generated by a polynomial whose zero lies in the open unit disk.

The following is main result of this section:

Proposition 7.8. *If $c(\chi) \notin \mathbb{Z}_{\geq 0}$, N_χ is a simple $L[[I]]$-module.*

Proof. Let I_f be the ideal of $L[[N_1']]$ which corresponds to an $L[[I]]$-submodule M of N_χ, where f is a polynomial which generates I_f.

Let t_x be the diagonal matrix

$$\begin{pmatrix} x & 0 \\ 0 & \bar{x}^{-1} \end{pmatrix}.$$

Then the action of t_x changes f into

$$\chi(t_x) f(\gamma^{N(x^{-1})} - 1).$$

Write $\omega_a(x) = (x+1)^a - 1$ and we could re-write the former as

$$t_x : f(\gamma - 1) \mapsto \chi(t_x) f(\omega_{N(x^{-1})}(\gamma - 1))$$

But the ideal I_f is stable under the above action by assumption. Hence, if z is a zero of f, $\omega_u(z)$ is also a zero of f for any $u \in \mathbf{Z}_p^\times$, which forces that $z+1$ must be a p^m-th root of unity for some $m \in \mathbb{N}$. Therefore, f is divisible (as a polynomial) by $\omega_{p^{m_0}}(x)^l$ for some $m_0, l \in \mathbb{N}$. In particular, when $k \geq m_0$ is large enough, the polynomial $\omega_{p^k}(x)^l$ lies in I.

Next, we look at the action of $n(\eta - \bar{\eta})$. We start with the following identity of matrices

$$n(\eta - \bar{\eta}) n'(np(\eta - \bar{\eta})) = n'(u_n^{-1} np(\eta - \bar{\eta})) \mathrm{diag}(u_n, u_n^{-1})$$

for $n \in \mathbb{Z}_{\geq 0}$, where u_n is the unit $1 + np(\eta - \bar{\eta})^2$ in $\mathbf{Z}_p^\times$. In $L[[N_1']]$, we have $n(\eta - \bar{\eta}) \cdot \gamma^n = \chi(\mathrm{diag}(u_n, u_n^{-1}))\gamma^{n/u_n}$. Hence,

$$n(\eta - \bar{\eta}) \cdot (\gamma^{p^k} - 1)^l = \textstyle\sum_{j=0}^{l} (-1)^{(l-j)} \binom{l}{j} \chi(\mathrm{diag}(u_{jp^k}, u_{jp^k}^{-1})) \gamma^{jp^k u_{jp^k}^{-1}}.$$

Combining that we have just described, we see when $k \geq m_0$, the polynomial $\omega_{p^k}(x)^l$ and its image under $n(\eta - \bar{\eta})$ both lie in the ideal I_f. We are certainly done if for some $k \geq m_0$ the two polynomials don't have common zeros, as in that case the ideal I_f would be the whole ring. On the other hand, one has, for any large enough k,

$$\textstyle\sum_{j=0}^{l} (-1)^j \binom{l}{j} \chi(\mathrm{diag}(u_{jp^k}, u_{jp^k}^{-1})) = 0,$$

i.e.,

$$\textstyle\sum_{j=0}^{l} (-1)^j \binom{l}{j} \exp(c(\chi) \cdot \log(1 + jp^k(\eta - \bar{\eta})^2)) = 0.$$

As a function (in variable y) analytic in a small ball around zero,

$$\sum_{j=0}^{l} (-1)^j \binom{l}{j} \exp(c(\chi) \cdot \log(1+jy))$$

has infinitely many zeros in that region which forces it to vanish. To complete the proof, we only need to show this is not possible when $c(\chi) \notin \mathbb{Z}_{\geq 0}$. Assume $c(\chi) \notin \mathbb{Z}_{\geq 0}$. Then a little inductive calculation of the higher derivatives of the former function and its values at zero gives:

$$\sum_{j=0}^{l} (-1)^j \binom{l}{j} j^m = 0, \text{ for any } m \in \mathbb{N}.$$

This is absurd. $\square$

Remark 7.9. *The limitedness of the above argument is obvious, as the nice property of $L[[N_1']]$ from our assumption is crucially used, which does not make sense in general.*

Corollary 7.10. *When $c(\chi) \notin \mathbb{Z}_{\geq 0}$, M_χ is a simple $L[[K]]$-module.*

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